QCD Correlation Functions of Heavy-light Hybrid Mesons ($J^P = 1^-$)

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Abstract

Quantum chromodynamics (QCD) predicts many bound states that have not yet been conclusively identified, and as more charmonium-like XYZ states are being discovered, interest is increasing in matching these theoretical bound states with experimental observation. Among these states are hybrid mesons: bound states of a quark, an antiquark, and a gluon. With upcoming experiments such as GlueX and PANDA, experimental data within the expected mass ranges of hybrids will be abundant in the next decade, and theoretical predictions are needed to help identify them. We calculate the correlation function associated with a heavy-light (open-flavour) $J^P = 1^-$ hybrid system, including non-perturbative condensate contributions up to six dimensions.

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To my wife, for not regretting the choice to marry a theoretical physicist.

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List of Abbreviations

LHS	Left-hand Side
LO	Leading Order
NLO	Next-to-leading Order
OPE	Operator Product Expansion
RHS	Right-hand Side
SSB	Spontaneous Symmetry Breaking
VEV	Vacuum Expectation Value
QCD	Quantum Chromodynamics
QFT	Quantum Field Theory

Chapter 1
Introduction

1.1 The Quark Model

Arguably, all of the physical sciences can, at various scales, be broken down into the investigation of the structure and interaction of matter. As history has progressed, science has delved deeper and deeper into the structure of matter, moving from the macroscopic to the molecular, from the molecular to the atomic, and from the atomic into the nuclear and subnuclear. At each step along the way we have discovered new structures; molecules can be broken down into atoms, which can be broken down into electrons, protons, and neutrons; these protons and neutrons can further be broken down into constituent fermions which we call quarks. There exists six varieties of quarks (called flavours) playfully named up, down, charm, strange, top, and bottom. Intertwined with the idea of the structure of matter is that of the interactions between matter, which is perhaps what makes our universe so interesting. Instead of particles being unaffected by their surroundings, they influence one another. All interactions can be reduced to one of four fundamental forces: gravity, electromagnetism, and the strong and weak forces. It is the latter three of these interactions that are best and most completely described by quantum field theories (in fact, the electromagnetic and weak forces are unified under electroweak theory). In this framework, force carriers (or more technically, gauge bosons) exchanged between particles are the cause for all interactions (the macroscopic and the microscopic) that we observe in everyday life. Electromagnetism is described by the exchange of photons, the weak nuclear force by the exchange of W and Z bosons, and the strong force (which is the interaction between quarks) is mediated by gluons. It is the strong force which we are primarily interested in.

It was a mathematical leap of faith that initially brought us the idea that particles such as the proton and neutron were composite particles made up of more fundamental pieces; Gell-Mann's [1] and Zweig's [2] postulation of quarks in the mid-sixties, through the use of group symmetries led to the definition of hadrons (particles made up of quarks) and their subclasses, distinguished by their internal structure: baryons (or antibaryons), which were made up of three quarks (or three antiquarks), and mesons, made up of a quark-antiquark pair. This became known as the quark model, and was the infancy of a formal field-theoretical description of the strong force. In order for a description of hadrons and their constituent quarks to be consistent with existing quantum theory (i.e., the Pauli exclusion principle), quarks were postulated to carry an additional quantum number called the colour charge made up of three "charges" often represented as the colours red, blue, and green (with corresponding anti-red, anti-blue, and anti-green charges) [1] [3, 4]. It turns out that the eight gluons are also colour charged, carrying a colour and an anti-colour. This concept of colour proves useful in providing a description of why no free quarks have ever been observed. All hadronic structures observed at present can be described as having no net colour; a neutron, for example, carries an up quark and two down quarks having a colour-neutral combination of colour charges. As individual quarks are coloured, they are not seen outside of bound states involving other coloured particles. Bound states of quarks must form colour singlet (or colourless) states. This phenomenon preventing the existence of free quarks is referred to as colour confinement, and has been a powerful tool in the prediction of hadronic particles.

1.2 Quantum Chromodynamics

Though the scientific community was hesitant to accept the idea of physical quarks, experimental evidence for their existence soon emerged from electron-proton scattering experiments in the 1960s [5]. These experiments produced results which suggested that the proton had an internal structure of three spin-$\frac{1}{2}$ particles which were loosely bound inside the proton at short distances. However, high energy collisions with the proton produced additional hadrons; none of the internal proton structure was broken loose. As well, the total momentum of electrically-charged particles did not completely account for overall total momentum,

[1]These colours have nothing to do with the electromagnetic spectrum.

suggesting some other electrically-neutral particles were involved (later identified as gluons). In building a theory to replicate these phenomena, three things needed to be accounted for:

1. the spin-$\frac{1}{2}$, electrically-charged particles in the internal structure
2. the loosely bound particles at high energies with confining properties
3. additional electrically-neutral particles as part of the internal structure to compensate for the missing momentum.

The loosely bound structure at high energies is a property referred to as asymptotic freedom [6]; simply put, contrary to the familiar forces of electromagnetism and gravitation, as separation between particles increases, the binding force increases as well. The confining properties that prevent the internal structure from being broken apart and viewed explicitly is explained by the previously discussed concept of colour singlet states. Additionally, as quantum electrodynamics (QED) and the electroweak theory describe interactions through the exchange of gauge bosons arising from gauge invariance, it seemed natural to look for a gauge-invariant theory which could offer a description of a corresponding electrically neutral gauge boson that might explain the discrepancy in momentum found in experiment. From here, the quantum field theory describing the phenomenon of the strong force was determined, known today as quantum chromodynamics (QCD). QCD is a generalization of QED, and carries a similar gauge invariance. It generalizes the $U(1)$ gauge symmetry of QED to an $SU(3)$ symmetry, which describes the colour charges predicted by Gell-Mann [1]. Though QCD is a generalization of QED, it is decidedly more intricate than its predecessor, becoming now a non-abelian theory with self-interacting gauge bosons. QCD is described through the Lagrangian given by[2]

$$\mathcal{L}_{\mathrm{QCD}}(x) = -\frac{1}{4}\left(G^a_{\mu\nu}(x)\right)^2 + \sum_A \bar{q}_A(x)\,(i\not{D} - m_A)q_A(x)\,, \tag{1.1}$$

where $G^a_{\mu\nu}(x) = \partial_\mu B^a_\nu(x) - \partial_\nu B^a_\mu(x) + g_s f^{abc} B^b_\mu(x)\, B^c_\nu(x)$ is the gluon field strength tensor, g_s is our coupling constant, $\bar{q}_A(x)$ and $q_A(x)$ are Dirac spinors representing antiquark and quark

[2]We use natural units where $c = \hbar = 1$, the fine-structure constant is $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$, and the strong coupling constant at the Z boson mass $m_Z = 91.2\,\mathrm{GeV}$ is $\alpha_s = \frac{g_s^2}{4\pi} \approx 0.1186$ [7].

fields (summed over flavour index A), $\not{D} = D^{\mu}\gamma_{\mu} = (\partial^{\mu} - ig_s t_a B_a^{\mu}(x))\gamma_{\mu}$ is the usual gauge-covariant derivative, and m_A is the mass for the corresponding flavour A. We have omitted the gauge-fixing and ghost terms in the Lagrangian above. Here, the index a represents gauge boson colour charge, while μ and ν represent the Lorentz structure, t_a are the generators in the fundamental representation of $SU(3)$, and $B_a^{\mu}(x)$ are gauge boson (gluon) vector fields. The $SU(3)$ symmetry group has the Lie algebra defined by the generators t^a

$$\left[t^a, t^b\right] = if^{abc}t^c, \tag{1.2}$$

where here (as in our definition of $G_{\mu\nu}^a(x)$ above), f^{abc} is a set of structure constants.

1.3 Hybrid Mesons

The quark model as described by Gell-Mann and Zweig provides a useful structure for predicting and organizing the bound states of quarks, and these hadronic structures for the most part are well-understood. But nowhere has it been stated that particles may only consist of either two or three quarks; the only condition placed on bound states is that they be colour singlets. Colour confinement permits us to formulate additional colourless bound states of quarks and gluons that have not been conclusively observed. With this in mind, many people have postulated more complicated structures. Why shouldn't bound states of four, five, or six quarks exist? As our gluons carry colour as well, it should be permitted in our current theoretical framework for quarks to form bound states with gluons, forming what are called hybrid mesons. Although there are several such proposed structures outside of the conventional quark model, the focus of our calculation is restricted to hybrid mesons.

Interest in hybrids has recently been rekindled due to a number of unexplained particles which have been observed over the past decade, but not yet classified. The $\pi_1(1400)$, $\pi_1(1600)$, and $\pi_1(2015)$ mesons ($J^{PC} = 1^{-+}$) are currently the most promising candidates for hybrid mesons, though experimental results are so far inconclusive [8]. With the construction of experiments designed to examine hybrid mesons, such as the GlueX experiment at Jefferson Lab [9] and the planned implementation of PANDA in Germany [10], experimental data within the expected mass ranges of hybrids will be abundant in the next decade, and there is a pressing need to expand phenomenological predictions. That these structures have not yet been observed is an outstanding problem in QCD; if hybrid mesons exist, then where are they? If they don't exist, then what are we missing in our construction of colour confinement? Finding hybrids would provide strong support for our characterization of colour confinement, and put an outstanding question in physics to rest. Further, the existence of hybrid mesons would provide us with a new hadronic system to study. On the other hand, not finding hybrids would open us up to improving upon our understanding of QCD. Either way, the presence of hybrids in QCD is an important question which must be answered if we are to fully understand the structure and formation of matter.

To confirm the identity of a particle resonance, theoretical results must match with experimental evidence. Some of the key identifiers are decay properties, mass, spin angular momentum (J), parity (P), and charge conjugation (C) quantum number. Of importance to us is the mass (which we intend to calculate) and the J, P, and C quantum numbers, collectively denoted as J^{PC}, which is determined at the outset of the calculation. In order to determine these quantum numbers, we must identify the orbital (L) and spin (S) angular momentum of the system. For a meson made up of two spin-$\frac{1}{2}$ constituents, $S = 0, 1$, and the value of L is dependent on its orbital quantum number where $L^2 = l(l+1)$. The J^{PC} for mesons are obtained through the results:

$$\vec{J} = \vec{L} + \vec{S} \tag{1.3}$$

$$P = (-1)^{L+1} \tag{1.4}$$

$$C = (-1)^{L+S}. \tag{1.5}$$

So, for a meson we can form J^{PC} states such as

$$J^{PC} \in \{0^{-+}, 0^{++}, 1^{--}, 1^{+-}, 1^{++}, 2^{--}, ...\}. \tag{1.6}$$

Based on the list above, there are specific allowed combinations of J^{PC} quantum numbers for mesons; in other words, for a traditional quark-antiquark mesonic structure, there is no way we may obtain states such as

$$J^{PC} \in \{0^{+-}, 0^{--}, 1^{-+}\}, \tag{1.7}$$

unless some additional degree of freedom is involved. Of interest to us are the open-charm D and D_s hybrid mesons, respectively bound states of a charm quark and light (up or down) antiquark, and a charm quark and strange antiquark (as well as their antiparticle configurations) with an additional gluonic degree of freedom. We call these sets of quantum numbers (which are inaccessible to conventional states) *exotic* quantum numbers. We are examining a specific state with $J^P = 1^-$; the structural nature of open-flavour systems is such that they are not eigenstates of the charge conjugation operator C.

1.4 Building Up the Field Theory

1.4.1 The Correlation Function

From quantum field theory, we are familiar with the two-point correlation function (otherwise known as a two point Green's function) which contains information on the propagation of a particle between two points in coordinate space, in addition to the bound-states and two-particle states in the interacting theory. A correlation function in a ϕ^4 interacting theory (i.e. a theory with an interaction Lagrangian term of $\mathcal{L}_{int} = \lambda\phi^4$) can be expressed as

$$\langle\Omega|\, T\phi(x)\phi(y)\, |\Omega\rangle \,, \tag{1.8}$$

where T is the time-ordering operator. Using composite operators, we can generalize this idea to bound states by defining a current representative of the properties of the particle of interest. In the case of hybrid mesons, we may define our currents subject to a few constraints. The current must

1. be a colour singlet, as all observed bound-states are "colourless"

2. exhibit the appropriate quark and gluon degrees of freedom, including the relevant flavour structure

3. contain the correct Lorentz structure to correspond to desired quantum numbers J^{PC}.

Given a current j_μ corresponding to a bound-state with the above constraints, we can write the correlation function

$$\langle\Omega|\, T j_\mu(x)\, j_\nu^\dagger(y)\, |\Omega\rangle\,. \tag{1.9}$$

We are interested in describing open-flavour hybrid states, specifically charm-light and charm-strange states. As we are handling the charm-strange states as an $\mathcal{O}\left(m_s\right)$ correction on the charm-light states, we can describe both particle states through a single current describing the hybrid meson structure, charm and strange flavour content, and the J^P quantum numbers, given by

$$j_\mu(x) = g_s\, \bar{c}_i^\alpha\, t^a_{\alpha\beta}\, \gamma^\rho_{ij}\, G^a_{\mu\rho}(x)\, s^\beta_j. \tag{1.10}$$

Note that while the individual current (1.10) is not Hermitian, the combination of the "creation" and "annihilation" current in (1.9) (i.e. $j_\mu(x)\, j_\nu^\dagger(y)$) is. From here we can define the correlation function in momentum space

$$\Pi_{\mu\nu}\left(q\right) = i\int d^dx\, e^{iq\cdot x}\langle\Omega|T j_\mu\left(x\right) j_\nu^\dagger\left(0\right)|\Omega\rangle, \tag{1.11}$$

where the $\langle\Omega|$ and $|\Omega\rangle$ states represent the vacuum as seen in the interacting theory. Here we perform the integral in an arbitrary number of d dimensions; this prepares the way for our dimensional regularization scheme which we will discuss in more detail in sections following. The form of the correlation function in (1.11) is the central quantity in our study of open-charm hybrid mesons. The correlation function itself may be separated into its scalar and vector components

$$\Pi_{\mu\nu}\left(q\right) = \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right)\Pi_v\left(q^2\right) + \frac{q_\mu q_\nu}{q^2}\Pi_s\left(q^2\right), \tag{1.12}$$

from which we can project out the state of interest, the vector ($J = 1$) state, $\Pi_v(q^2)$ by applying an orthogonal projector $\frac{1}{(d-1)}\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right)$ to both sides, giving

$$\frac{1}{(d-1)}\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right)\Pi_{\mu\nu}\left(q\right) = \Pi_v\left(q^2\right), \tag{1.13}$$

where $g_{\mu\nu}$ is the metric tensor (which we will explicitly define in Chapter 2), and we have expressed the projection operator in d dimensions.

1.4.2 Wick's Theorem

In order to evaluate (1.11) which is phrased in terms of the ground state of the interacting theory, recall from field theory that the interacting theory may be expressed in terms of the free theory (where the free theory vacuum is denoted by $|0\rangle$). In terms of a general field ψ (with corresponding free fields ψ_0),

$$\langle\Omega|\, T\psi(x)\,\overline{\psi}(y)\,|\Omega\rangle = \frac{\langle 0|\, T\psi_0(x)\,\overline{\psi_0}(y)e^{i\int d^4\mathrm{x}\mathcal{L}_{\mathrm{int}}[\psi_0]}\,|0\rangle}{\langle 0|\, Te^{i\int d^4\mathrm{x}\mathcal{L}_{\mathrm{int}}[\psi_0]}\,|0\rangle}. \tag{1.14}$$

Given a form for the interaction Lagrangian, $\mathcal{L}_{\mathrm{int}}$, we can perturbatively expand (1.14) in the strong coupling constant g_s to evaluate the correlation function in the interacting theory. Now that we have phrased our problem in terms of a correlation function of quantum fields, we may apply Wick's theorem to evaluate the expectation value. Wick's theorem relates the time-ordered products preserving causality to what are called *normal-ordered products* which are defined such that all creation operators are commuted towards the bra vectors, and all the annihilation operators are commuted towards the ket vectors. Considering general fermionic fields $\psi(x)$, we can relate the time-ordered and normal-ordered product of operators by introducing the *contraction* of two fields, where this contraction can be defined as

$$\overbracket{\psi(x)\overline{\psi}}(y) \equiv iS(x-y) \equiv \langle 0|\, T\left(\psi(x)\overline{\psi}(y)\right)|0\rangle = \int \frac{d^4\mathrm{p}}{(2\pi)^4}\frac{i(\not{p}+m)}{p^2-m^2+i0^+}e^{-ip\cdot(x-y)}, \tag{1.15}$$

where we can relate this contraction to the fermion propagator $S\,(x-y)$. Wick's Theorem can be stated in terms of arbitrarily many bosonic fields as[3]

$$T\{\phi(x_1)\phi(x_2)\cdots\phi(x_m)\} = N\{\phi(x_1)\phi(x_1)\cdots\phi(x_m) + \text{all possible contractions}\}. \tag{1.16}$$

[3]The proof for this may be found in [11]

Note that the alternate notation $:\psi(y)\overline{\psi}(x):$ is often used to denote normal-ordering as well. For fermionic fields, the only addition to (1.16) would be appropriate factors of (-1) to account for Fermi statistics. These contractions are a consequence of the non-commutative nature of the ladder operators used to define the fields [11], or equivalently due to the algebra of the fields themselves. Expressing our time-ordered product in terms of normal-ordered products allows us to simplify our calculation in terms of Feynman propagators resulting from these contractions. Let us consider an explicit example, the time-ordered product of four interacting bosons. Applying (1.16), we obtain

$$\begin{aligned}
T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} = N\{&\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) + \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \\
&+ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) + \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \\
&+ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) + \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \\
&+ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) + \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \\
&+ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) + \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} \\
= N\{&\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} + D(x_1 - x_2)N\{\phi(x_3)\phi(x_4)\} \\
&+ D(x_1 - x_3)N\{\phi(x_2)\phi(x_4)\} + D(x_1 - x_4)N\{\phi(x_2)\phi(x_3)\} \\
&+ D(x_2 - x_3)N\{\phi(x_1)\phi(x_4)\} + D(x_2 - x_4)N\{\phi(x_1)\phi(x_3)\} \\
&+ D(x_3 - x_4)N\{\phi(x_1)\phi(x_2)\} + D(x_1 - x_2)D(x_3 - x_4) \\
&+ D(x_1 - x_3)D(x_2 - x_4) + D(x_1 - x_4)D(x_2 - x_3),
\end{aligned} \tag{1.17}$$

where D is the propagator resulting from the contractions of boson field operators. Notice that the final expression has non-contracted normal-ordered products remaining. In general, the vacuum expectation value (VEV) of a normal-ordered product is zero, i.e.

$$\langle 0|\, N\{\phi(x_1)\phi(x_2)\cdots\phi(x_m)\}\,|0\rangle = 0 \tag{1.18}$$

due to the properties of $|0\rangle$. This holds in most situations, but QCD is a specific exception with several non-zero vacuum expectation values appearing. Through theory and experiment, we have good evidence that these VEVs are non-zero; they are an indication that spontaneous symmetry breaking is taking place and reflect the complexity of the QCD vacuum state.

1.4.3 Condensates and Spontaneous Symmetry Breaking

Spontaneous symmetry breaking occurs when the Lagrangian of a given theory holds a certain symmetry which is not exhibited by the vacuum; a classic example is that of ferromagnetism. Above the Curie temperature, the alignment of the spins of a material's constituent electrons are random and unaligned; there exists a rotational symmetry as the material produces no overall magnetic field, and no particular direction is singled-out. Below the Curie temperature, the spins of the electrons are coupled, and produce an overall magnetic dipole. Here, a particular direction is now distinguished and the symmetry has been "broken", even though the Lagrangian still carries rotational symmetry. More formally, spontaneous symmetry breaking can be stated as:

$$Q^0 \left|0\right\rangle \neq 0, \tag{1.19}$$

where Q^0 represents a conserved charge associated with a Noether current generated from a symmetry in the associated Lagrangian. This is an indication of broken symmetry between the QCD Lagrangian and the ground state of the theory (in this case, the vacuum). These non-zero VEVs, also known as condensates, are an indication of a complicated, dynamical QCD vacuum, contrary to the notion of the vacuum being empty space. They represent non-perturbative contributions to the correlation function. In QCD, it is the *chiral symmetry* that is broken. Consider for a moment the two lightest quarks (u, d), and the QCD Lagrangian that governs them:

$$\begin{aligned}\mathcal{L}_{QCD} &= -\frac{1}{4}(G^a_{\mu\nu})^2 + i\bar{u}^L \not{D} u^L + i\bar{u}^R \not{D} u^R + i\bar{d}^L \not{D} d^L + i\bar{d}^R \not{D} d^R - m_u \bar{u}u - m_d \bar{d}d \\ &\approx -\frac{1}{4}(G^a_{\mu\nu})^2 + i\bar{u}^L \not{D} u^L + i\bar{u}^R \not{D} u^R + i\bar{d}^L \not{D} d^L + i\bar{d}^R \not{D} d^R.\end{aligned} \tag{1.20}$$

Equation (1.20) is phrased in terms of chiral spinors, where L and R notate the left- and right-handed spinors respectively, and we have also applied the assumption that $m_u \approx m_d \approx 0$. In this massless quark model, we have the symmetries

$$\begin{pmatrix} u^{L,R} \\ d^{L,R} \end{pmatrix} \rightarrow g_{L,R} \begin{pmatrix} u^{L,R} \\ d^{L,R} \end{pmatrix} \tag{1.21}$$

where $g_{L,R}$ represents separate rotations under $SU(2)$ representing symmetries reflected in our massless Lagrangian (1.20); that massless Lagrangian carries an overall symmetry of $U(2) \times U(2) = SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A$ (the V and A denote vector and axial symmetries). The chiral symmetry which is spontaneously broken is represented by the $SU(2)_L \times SU(2)_R$ symmetry, a series of continuous flavour rotations. With this symmetry exposed, and with the help of Noether's theorem, we can extract conserved currents from the $SU(2)$ symmetries in the Lagrangian.[4] A consequence of spontaneous symmetry breaking is the generation of Nambu-Goldstone bosons. The Nambu-Goldstone Theorem [12] states that for each degree of freedom of spontaneously-broken continuous symmetry, a massless bosonic state is generated. In our example of ferromagnetism, below the Curie temperature the "spin waves" (propagation of perturbations in the correlated electron spins) are the massless bosons created. This production of Nambu-Goldstone bosons is what motivates our pursuit of symmetry breaking in QCD.

Table 1.1: Summary of Meson Masses for u, d, and s Flavour Content

Meson	Quark Content	Mass (MeV)
π^0	$\frac{u\overline{u}-d\overline{d}}{\sqrt{2}}$	135
π^+	$u\overline{d}$	140
π^-	$d\overline{u}$	140
K^0	$d\overline{s}$	498
$\overline{\mathrm{K}}^0$	$s\overline{d}$	498
K^+	$u\overline{s}$	494
K^-	$s\overline{u}$	494
η	$\frac{u\overline{u}+d\overline{d}-2s\overline{s}}{\sqrt{6}}$	548
η'	$\frac{u\overline{u}+d\overline{d}+s\overline{s}}{\sqrt{3}}$	958

Consider the listed masses of some of the mesons in our quark model (Table 1.1). The

[4]This $SU(2)$ symmetry can be extended to include all three light quarks (u, d, and s) in an $SU(3)$ symmetry. The strange quark mass may be considered nearly massless compared against the QCD scale parameter, $\Lambda_{QCD} \approx 300\,\mathrm{MeV}$

constituent quarks that make up each of the bound states are small with $m_u = 2.3\,\text{MeV}$, $m_d = 4.8\,\text{MeV}$, and $m_s = 95\,\text{MeV}$ [13]. Even with the discrepancies in masses, the family of pions (π^0, π^+, π^-) are significantly lighter than the other bound states. In the light of spontaneous symmetry breaking, this begins to make sense. We see the $SU(2)$ chiral symmetry present in the massless QCD Lagrangian, and the breaking of a continuous symmetry such as $SU(2)$ should generate massless Nambu-Goldstone bosons. However due to the non-zero mass carried by the light quarks, the symmetry breaking is only approximate, and as such pseudo-Nambu-Goldstone bosons are generated, which is consistent with the anomalously light yet massive pions. Associated with this broken symmetry is the mass-dimension[5] four chiral condensate, $m\langle \overline{q}q\rangle$. However, other condensates also appear at the non-perturbative level; at dimension four, we also have the gluon condensate $\langle \alpha G^2\rangle$ associated with the breaking of scale-invariance. At higher mass dimensions we have the dimension five "mixed" condensate $\langle \overline{q}\sigma \cdot Gq\rangle$ and the dimension six gluon condensate, $\langle g^3 G^3\rangle$, where they are defined as

$$\langle \alpha G^2\rangle = \alpha_s \langle G^a_{\mu\nu}(0) G^a_{\mu\nu}(0)\rangle \tag{1.22}$$

$$\langle \overline{q}q\rangle = \langle \overline{q}^\alpha_i(0) q^\alpha_i(0)\rangle \tag{1.23}$$

$$\langle \overline{q}\sigma \cdot Gq\rangle = i g_s t^a_{\alpha\beta} \left(\sigma^{\mu\nu}\right)_{ij} \langle \overline{q}^\alpha_i(0) G^a_{\mu\nu}(0) q^\beta_j(0)\rangle \tag{1.24}$$

$$\langle g^3 G^3\rangle = g_s^3 \langle f^{abc} G^a_{\alpha\beta}(0) G^b_{\beta\gamma}(0) G^c_{\gamma\alpha}(0)\rangle, \tag{1.25}$$

where $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu \gamma^\nu\right]$ in equation (1.24). These higher-ordered condensates may be related to the more fundamental dimension four quark and gluon condensates associated with broken symmetries. Although a full discussion of the numerical values of these condensates is beyond the scope of this work, the best known condensates values are the quark condensate [14],

$$(m_u + m_d)\langle \overline{u}u + \overline{d}d\rangle = -2m_\pi^2 (92.4\,\text{MeV}), \tag{1.26}$$

[5]We refer to condensates in our work by their dimensionality as it pertains to the order of expansion in the operator product expansion discussed in the next section.

(where m_π is the mass of the pion), and the gluon condensate [15],

$$\langle \alpha G^2 \rangle = (7.1 \pm 0.9) 10^{-2}\,\mathrm{GeV}^4. \tag{1.27}$$

1.4.4 The Operator Product Expansion

In forming our correlation function (1.11), we utilize composite operators such as the current in (1.10), where these currents act at a particular spacetime coordinate. In cases such as these, we can use a tool called the *operator product expansion* in our calculation of (1.11). The operator product expansion states that for an operator $\mathcal{O}$ acting at a spacetime coordinate x,

$$\int d^4 x e^{iq\cdot x} T\left(\mathcal{O}(x)\mathcal{O}(0)\right) = \sum_n C_n\left(q\right)\mathcal{O}_n\left(0\right). \tag{1.28}$$

This relationship states that the information expressed in the position-dependent non-local operators may be shifted into momentum dependent coefficients $C_n(q)$. If we keep in mind that we may evaluate (1.11) using (1.16), writing our correlation function in terms of normal-ordered local operators, we see that the contractions start to make up our coefficients $C_n(q)$ (once transformed into the appropriate momentum space). The spacetime dependence in our non-local VEVs can thus be shifted into the coefficients, which are just numbers, and not dependent on the external particle state in any way. In this way, the large distance and short distance effects can be separated into the coefficients $C_n(q)$ and the operators $\mathcal{O}_n\left(0\right)$, respectively. Specifically, this allows us to expand our correlation function in terms of local vacuum expectation values. We calculate the OPE of a heavy-light hybrid meson system with $J^P = 1^-$, including up to the dimension six gluon condensate, $\langle g_s G^3 \rangle$.

1.4.5 Regularization and Renormalization

Integral Divergences

One of the obstacles encountered in the development of formal quantum field theory was the appearance of divergent integrals in physically-finite quantities, such as the charge of the electron. One common example of these divergences is in the integration of one-loop integrals of the form

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i0^+)^n}. \tag{1.29}$$

This integral is divergent for $n = 1, 2$. The solution proposed to deal with these divergences is two-fold: first a regularization procedure is applied to the integral to isolate the divergent terms in the integral, and then a renormalization scheme is applied to extract the physical contributions from the finite terms, and manage the divergences.

Dimensional Regularization

There are a few methods available to regularize these divergences, however in our consideration of QCD we will find *dimensional regularization* ("dim-reg") to be the most concise and practical method. One of the useful qualities of dim-reg as a regularization scheme is that it preserves the symmetries (in this case specifically, the gauge symmetry) of the Lagrangian. In practice, if we consider our previous integral (1.29) and promote the integral to an arbitrary d spacetime dimensions,

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i0^+)^n} \rightarrow \nu^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - \Delta + i0^+)^n}. \tag{1.30}$$

Here, ν is a mass scale introduced to compensate dimensionally for the promotion of $d^4k \rightarrow d^dk$. We let $d = 4 + 2\epsilon$ (a convention consistent with [16]), Euclideanize our coordinates, and evaluate our integral in d-dimensional spherical coordinates, where

$$\int d^dk = \int d\Omega_d \int k^{d-1} dk = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \int k^{d-1} dk, \tag{1.31}$$

with the integral now being expressed over the magnitude of the radial coordinate k. This leads us to the final result

$$\nu^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - \Delta + i0^+)^b} = i(-1)^{-b} \frac{1}{(4\pi)^{d/2}} \frac{\nu^{4-d}}{\Delta^{b-\frac{d}{2}}} \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(b - \frac{d}{2}\right)}{\Gamma(b)\,\Gamma\left(\frac{d}{2}\right)}. \tag{1.32}$$

Implementing a modified minimal subtraction renormalization scheme (otherwise known as $\overline{\mathrm{MS}}$ scheme), we rescale our renormalization parameter redefining

$$\nu^2 \to \nu^2 \left(\frac{e^{-\gamma_E}}{4\pi} \right), \tag{1.33}$$

where γ_E is Euler's constant.

1.4.6 TARCER and the Evaluation of Integrals

In order to calculate two-loop integrals appearing in the contributions to the correlation function, we utilize recurrence relations implemented in *Mathematica* through the program TARCER [17] to exchange complicated numerator algebra in favor of additional powers of denominator factors. This reduces the necessary integrals to the following three, expressed in notation consistent with TARCER:

$$\begin{aligned}
\mathrm{TAI}[d,\,0,\,\{\{\nu_1,\,m_1\}\}] =& \frac{1}{\pi^{d/2}} \int \frac{d^d k_1}{\left(k_1^2 - m_1^2\right)^{\nu_1}} \\
\equiv& \mathbf{A}^d_{\{\nu_1,\,m_1\}} \\
\mathrm{TBI}[d,\,q^2,\,\{\{\nu_1,\,m_1\},\,\{\nu_2,\,m_2\}\}] =& \frac{1}{\pi^{d/2}} \int \frac{d^d k_1}{\left(k_1^2 - m_1^2\right)^{\nu_1} \left((k_1-q)^2 - m_2^2\right)^{\nu_2}} \\
\equiv& \mathbf{B}^d_{\{\nu_1,\,m_1\},\{\nu_2,\,m_2\}} \\
\mathrm{TJI}[d,\,q^2,\,\{\{\nu_1,\,m_1\},\,\{\nu_2,\,m_2\},\,\{\nu_3,\,m_3\}\}] =& \frac{1}{\pi^{d}} \int \frac{d^d k_1 d^d k_2}{\left(k_1^2 - m_1^2\right)^{\nu_1} \left((k_1-k_2)^2 - m_2^2\right)^{\nu_2}} \\
&\times \frac{1}{\left((k_2-q)^2 - m_3^2\right)^{\nu_3}} \\
\equiv& \mathbf{J}^d_{\{\nu_1,\,m_1\},\{\nu_2,\,m_2\},\{\nu_3,\,m_3\}}.
\end{aligned} \tag{1.34}$$

These remaining "master" integrals are then evaluated by applying results reported in the literature [16, 18, 19, 20]. We list the results specific to our calculation:

$$\mathbf{A}^d_{\{1,\,m\}} = -i \left(m^2\right)^{\frac{d}{2}-1} \Gamma\left(1 - \frac{d}{2}\right). \tag{1.35}$$

$$\mathbf{B}^{d}_{\{\beta,\,m\},\{\alpha,\,0\}} = i(-1)^{-\beta-\alpha}\left(m^2\right)^{\frac{d}{2}-\beta-\alpha}\frac{\Gamma\left(\frac{d}{2}-\beta\right)\Gamma\left(\beta+\alpha-\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\alpha\right)}\,{}_2F_1\left[\begin{matrix}\beta,\ \beta+\alpha-\frac{d}{2} \\ \frac{d}{2}\end{matrix}\Big|\frac{q^2}{m^2}\right]. \tag{1.36}$$

$$\begin{aligned}\mathbf{J}^{d}_{\{\nu,\,m\},\{1,\,0\},\{1,\,0\}} =& (-1)^{1-\nu}(m^2)^{d-2-\nu}\frac{\Gamma\left(\frac{d}{2}-1\right)^2\Gamma\left(\nu+2-d\right)\Gamma\left(2-\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\nu\right)} \\ &\times {}_2F_1\left[\begin{matrix}2-\frac{d}{2},\ \nu+2-d \\ \frac{d}{2}\end{matrix}\Big|\frac{q^2}{m^2}\right].\end{aligned} \tag{1.37}$$

Equation (1.37) is a result obtained by combining results from [19] and [16] by using a few change of variables, where the hypergeometric function shown is defined as

$${}_2F_1\left[\begin{matrix}a,\ b \\ c\end{matrix}\Big|z\right] = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}. \tag{1.38}$$

The so-called Pochhammer symbol in (1.38) can be expressed in terms of Gamma functions as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{1.39}$$

The TAI class of one-loop integrals are often referred to as "massive tadpole" integrals due to the geometry of their associated Feynman diagrams (Figure 1.1).

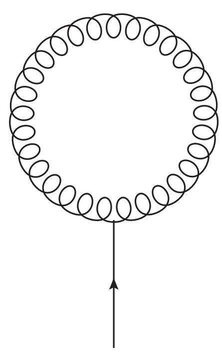

Figure 1.1: Example of a "massless tadpole" diagram

In the massless limit, these massive tadpoles become *massless* tadpoles, leading to the useful identity [16]

$$\mathrm{TAI}[d,\,q^2,\,\{\{\nu_1,\,0\}\}] \equiv \mathbf{A}^{d}_{\{\nu_1,\,0\}} = \frac{1}{\pi^{d/2}}\int\frac{d^dk_1}{\left(k_1^2\right)^{\nu_1}} = 0. \tag{1.40}$$

After calculating the perturbative and non-perturbative contributions forming the correlation function, we can apply QCD sum rules to calculate the hadronic mass.

1.5 QCD Sum Rules

QCD sum rules [21, 22] is one method of extracting ground state masses of particles, and is the focus of our calculation. The method begins with a dispersion relation satisfied by the components of the correlation function (1.12):

$$\Pi_{v,s}\left(Q^2\right) = \frac{1}{\pi}\int_{t_{\min}}^{\infty} ds\, \frac{\operatorname{Im}\Pi_{v,s}(s)}{s+Q^2} + \text{polynomials}. \tag{1.41}$$

Note that equation (1.41) is defined in Euclidean spacetime with $Q^2 = -q^2$ through a Wick rotation of Minkowskian momentum q. The above dispersion relation connects quarks on the left-hand side to hadronic states on the right. The left-hand side $\Pi\left(Q^2\right)$ is what we calculate perturbatively from QCD (with additional non-perturbative contributions), while the right-hand side contains (through $\frac{1}{\pi}\mathrm{Im}\Pi$, the hadronic spectral function) hadronic resonances that could be observed experimentally. The non-perturbative contributions (vacuum condensates) contain information on the influence of the vacuum on the hybrid system. In practice, we work with a Borel transform of (1.41) defined through the operator

$$\hat{\mathcal{B}} = \lim_{n\to\infty} \frac{1}{\Gamma(n)}\left(-Q^2\right)^n \left(\frac{d}{dQ^2}\right)^n,\ Q^2 \to \infty,\ \frac{n}{Q^2} \equiv \tau, \tag{1.42}$$

where τ is the *Borel parameter*. The Borel transform simplifies (1.41) by eliminating constant terms and polynomials in Q^2, and accentuates the low-energy (i.e. ground state) contribution to the right-hand side. The following identity from [16] is useful in applying the Borel transform to (1.41):

$$\hat{\mathcal{B}}\left[\frac{1}{(Q^2+s)^\beta}\right] = \frac{1}{\Gamma(\beta)}\tau^\beta e^{-s\tau}. \tag{1.43}$$

With this result, our dispersion relation (1.41) becomes

$$\hat{\mathcal{B}}\left[\Pi_{v,s}\left(Q^2\right)\right] = \frac{\tau}{\pi}\int_{t_{\min}}^{\infty} ds\, \mathrm{Im}\,\Pi_{v,s}\left(s\right) e^{-s\tau}. \tag{1.44}$$

Examining this equation, we see that the Borel transform may be related to the inverse Laplace transform by

$$\frac{1}{\tau}\hat{\mathcal{B}}\left[F\left(Q^2\right)\right] = \mathcal{L}^{-1}\left[F\left(Q^2\right)\right]. \tag{1.45}$$

The inverse Laplace transform is expressed as an integral

$$\mathcal{L}^{-1}\left[F\left(Q^2\right)\right] = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} dQ^2\, e^{-\tau Q^2} F\left(Q^2\right). \tag{1.46}$$

The convenience that the inverse Laplace transform expression of the Borel transform affords is trading the evaluation of a differential operator for a complex contour integration.

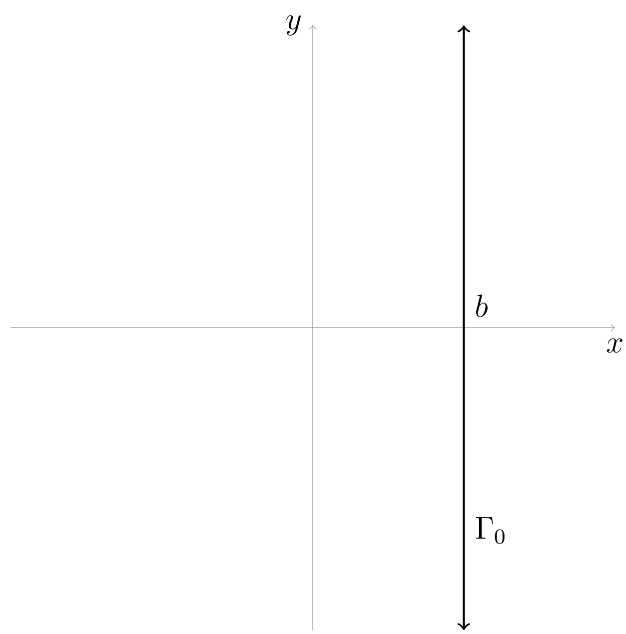

Figure 1.2: Integration contour for inverse Laplace transform

Consider the contour of integration Γ_0 for (1.46) shown in Figure 1.2. To evaluate this integral, we may deform this contour into the closed "keyhole" contour depicted in Figure 1.3, provided we take the limit $R \to \infty$.

We can then break the closed contour into individual pieces, and using Cauchy's Integral

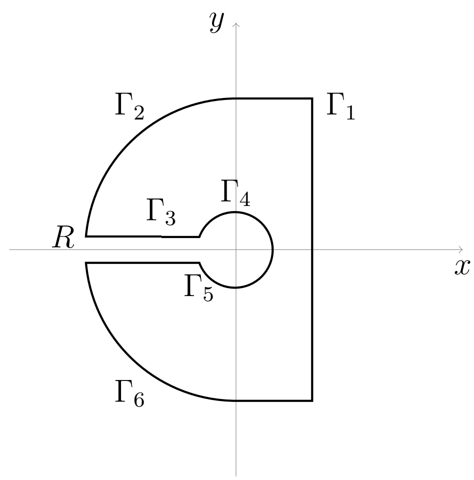

Figure 1.3: Deformed contour

Theorem,

$$\oint_{\Gamma_{\text{Total}}} \equiv \sum_{i=1}^{6} \int_{\Gamma_i} = 0. \tag{1.47}$$

By taking the limit $R \to \infty$ in the closed contour, we see that we may relate our original contour to our closed deformed contour by

$$\int_{\Gamma_0} = \lim_{R\to\infty} \int_{\Gamma_1}, \tag{1.48}$$

leaving us with a way to relate Γ_0 to (1.47),

$$\int_{\Gamma_0} = \lim_{R\to\infty} \left(-\sum_{i=2}^{6} \int_{\Gamma_i} \right). \tag{1.49}$$

We will find that the only contributions from the right side of (1.49) are associated with the branch and pole structure of our operator product expansion of the correlation function.

If we return to equation (1.44) and now examine the right-hand side, we see that we need information about $\operatorname{Im}\Pi(s)$ to evalute the integral. This expression is often referred to as the *hadronic spectral function*. To relate the quark dynamics contained in the QCD correlator to the dispersion relation describing the hadronic bound states, we must specify a model for the hadronic spectral function *a priori*. We are interested in the ground state mass of the hybrid bound state; experimental mass spectra show bound states as peaks centred on energies corresponding to the particle mass. We model the hadronic spectral function with what is called the *narrow resonance model*,

$$\operatorname{Im}\Pi(s) = \pi \sum_{r=1}^{n} f_r^2 m_r^2 \delta\left(s - m_r^2\right) + \theta\left(s - s_0\right) \operatorname{Im}\Pi^{\mathrm{QCD}}(s) \tag{1.50}$$

where $\operatorname{Im}\Pi^{\mathrm{QCD}}(s)$ represents the QCD continuum beginning at a value s_0; this is obtained from the complex structure of our correlation function. Specifically we only look at the first resonance ($n = 1$), as we are interested in the ground state of these mesons. With this model in place, (1.44) becomes

$$\mathcal{L}^{-1}\left[\Pi_{V,S}\left(Q^2\right)\right] = f_1^2 m_1^2 e^{-m_1^2 \tau} + \frac{1}{\pi}\int_{s_0}^{\infty} ds\, \operatorname{Im}\Pi^{\mathrm{QCD}}\left(s\right) e^{-s\tau}. \tag{1.51}$$

To determine the mass of our hadronic state, we define our Laplace sum rule as

$$R_0(\tau, s_0) = \mathcal{L}^{-1}\left[\Pi_{v,s}\left(Q^2\right)\right] - \frac{1}{\pi}\int_{s_0}^{\infty} ds\, \operatorname{Im}\Pi^{\mathrm{QCD}}\left(s\right) e^{-s\tau}, \tag{1.52}$$

and evaluate the left side of the following expression, giving the calculated mass of the hadronic state,

$$\frac{-\frac{d}{d\tau} R_0(\tau, s_0)}{R_0(\tau, s_0)} = m_H^2. \tag{1.53}$$

where values of τ and s_0 are numerically fitted.

Chapter 2

Calculation of the Correlation Function

2.1 Conventions, Definitions, and Miscellaneous Results

Primarily our work follows the conventions of [16], however we will review our notations and conventions before proceeding further. As previously mentioned in Chapter 1, we utilized "natural units" for simplified calculations ($\hbar = c = 1$). Field theoretical calculations take place in the following Minkowskian spacetime metric:

$$g_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.1}$$

QCD is a gauge theory with force-carrying gauge bosons; it is necessary to specify our gauge-covariant derivative,

$$D_\mu(x) = \partial_\mu - ig_s t^a B^a_\mu(x). \tag{2.2}$$

The gauge field $B^a_\mu(x)$ is a depiction of our force carriers, the gluons, while the group generator t^a contains their colour structure. For the purposes of our analysis, we primarily consider strange quark and antiquark fields ($s^\alpha_i(x)$, $\overline{s}^\alpha_i(x)$), and charm quark and antiquark fields ($c^\alpha_i(x)$, $\overline{c}^\alpha_i(x)$), decorated with quark colour α and Dirac index i. We denote the gluon field as $B^a_\mu(x)$ with gluon color index a, Lorentz index μ, acting at spacetime coordinate x. It is convenient to form the *gluon field strength tensor* in order to formulate a gauge-invariant

Langrangian,

$$G^a_{\mu\rho}(x) = \partial_\mu B^a_\rho(x) - \partial_\rho B^a_\mu(x) + g_s f^{abc} B^b_\mu(x) B^c_\rho(x). \tag{2.3}$$

Notice that this closely resembles the electromagnetic field strength tensor, with the exception of the interaction term introduced by the non-abelian nature of QCD. The gluon field strength tensor may be written as a commutator of the gauge-covariant derivative (2.2),

$$-\left[D_\mu,\, D_\nu\right] = i g_s t^a G^a_{\mu\nu}(x) \equiv G_{\mu\nu}(x), \tag{2.4}$$

(it can be convenient to represent this colourless quantity using the notation $G_{\mu\nu}(x)$ as shown). Additionally, we will perform our calculations of the non-local VEVs by utilizing the fixed-point gauge relating the gluon field to the gluon field strength tensor,

$$B^a_\mu(x) = \int_0^1 d\alpha\, \alpha G^a_{\omega\mu}(\alpha x) x^\omega. \tag{2.5}$$

where expanding around $x = 0$,

$$B^a_\mu(x) = \frac{1}{2} x^{\omega_0} G^a_{\omega_0\mu}(0) + \frac{1}{3} x^{\omega_0} x^{\omega_1} \partial_{\omega_1} G^a_{\omega_0\mu}(0) + \ldots. \tag{2.6}$$

Note that (2.6) implies that $B^a_\mu(0) = 0$. It has been shown in [23] that the utilization of the fixed-point gauge, which itself violates translation invariance, does not conflict with the covariant gauge in QCD.

As discussed in Chapter 1, the mass calculation of our hybrid system begins with a hybrid current (and its conjugate) associated with our particular open-charm system for a $J^P = 1^-$.

$$\begin{aligned} j_\mu(x) =& g_s \bar{c}^\alpha_i(x) t^{\alpha\beta}_a \gamma^\rho_{ij} G^a_{\mu\rho}(x) s^\beta_j(x) \\ j^\dagger_\mu(x) =& g_s \bar{s}^\alpha_i(x) t^{\alpha\beta}_a \gamma^\rho_{ij} G^a_{\mu\rho}(x) c^\beta_j(x) \end{aligned} \tag{2.7}$$

With this current appropriately describing the charm, strange, and gluonic content, we can analyze this hybrid system in our correlator (1.11). The QCD Lagrangian in its full form, restricted to the quark flavours involved in our calculation and including the Fadeev-Popov ghosts is

$$
\begin{aligned}
\mathcal{L}_{\rm QCD}(x) = &- \frac{1}{2}\left[\partial_\mu B_\nu^a(x)\right]\left[\partial^\mu B_a^\nu(x) - \partial^\nu B_a^\mu(x)\right] - \frac{1}{2a}\left[\partial_\mu B_a^\mu(x)\right]\left[\partial_\nu B_a^\nu(x)\right] \\
&+ \frac{i}{2}\left(\overline{s}_\alpha(x)\partial\!\!\!/ s_\alpha(x) + \overline{c}_\alpha(x)\partial\!\!\!/ c_\alpha(x)\right) - \frac{i}{2}\left(\left[\partial_\mu\overline{s}_\alpha(x)\right]\partial^\mu s_\alpha(x) + \left[\partial_\mu\overline{c}_\alpha(x)\right]\partial^\mu c_\alpha(x)\right) \\
&- m_{\rm s}\overline{s}_\alpha(x)s_\alpha(x) - m_{\rm c}\overline{c}_\alpha(x)c_\alpha(x) \\
&+ \frac{1}{2}g_s\left(\overline{s}_\alpha(x)\lambda^a_{\alpha\beta}\gamma^\mu s_\beta(x)B_\mu^a(x) + \overline{c}_\alpha(x)\lambda^a_{\alpha\beta}\gamma^\mu c_\beta(x)B_\mu^a(x)\right) \\
&- \frac{1}{2}g_s f_{abc}\left[\partial_\mu B_\nu^a(x) - \partial_\nu B_\mu^a(x)\right]B_b^\mu(x)B_c^\nu(x) \\
&- \frac{1}{4}g_s^2 f_{abc}f_{ade}B_\mu^b(x)B_\nu^c(x)B_d^\mu(x)B_e^\nu(x) \\
&- \left[\partial_\mu\overline{\phi_a}(x)\right]\partial^\mu\phi_a(x) + g_s f_{abc}\left[\partial_\mu\overline{\phi_a}(x)\right]\phi_b(x)B_c^\mu(x).
\end{aligned}
\tag{2.8}
$$

The diagrams at leading order in our calculation do not have the necessary topology to require the consideration of ghost particles, and terms of $\mathcal{O}\left(g_s^2\right)$ in $\mathcal{L}_{\rm int}$ (which are contributions to higher-order diagrams) are neglected. With this, we can separate the interacting Lagrangian at order $\mathcal{O}(g_s)$ into

$$
\mathcal{L}_{\rm int}(x) = \frac{1}{2}g_s\left(\left(\overline{s}_\alpha(x)\lambda^a_{\alpha\beta}\gamma^\mu s_\beta(x)B_\mu^a(x) + \overline{c}_\alpha(x)\lambda^a_{\alpha\beta}\gamma^\mu c_\beta(x)B_\mu^a(x)\right) - f_{abc}\underline{G}^a_{\mu\nu}(x)B_b^\mu(x)B_c^\nu(x)\right). \tag{2.9}
$$

Out of convenience, we have defined above $\underline{G}^a_{\mu\rho}(x) = \partial_\mu B^a_\rho(x) - \partial_\rho B^a_\mu(x)$ such that (2.3) is simplified to

$$
G^a_{\mu\rho}(x) = \underline{G}^a_{\mu\rho}(x) + g_s f^{abc}B^b_\mu(x)B^c_\rho(x). \tag{2.10}
$$

This cleans up (2.9), and will serve to simplify the calculation of the correlation function. Our previous expression (1.14) allowed us to express the interacting theory in terms of a perturbative expansion of the free theory with the interacting Lagrangian. Expressed in terms of our correlation function (1.11) and taking our perturbative expansion to next-to-leading order, we obtain

$$
\Pi_{\mu\nu}\left(q\right) = i\int d^4x\, e^{iq\cdot x}\langle\Omega|Tj_\mu\left(x\right)j_\nu^\dagger\left(0\right)|\Omega\rangle - \int d^4x\, d^4y\, e^{iq\cdot x}\langle\Omega|Tj_\mu\left(x\right)\mathcal{L}_{\rm int}(y)j_\nu^\dagger\left(0\right)|\Omega\rangle, \tag{2.11}
$$

which is evaluated utilizing the field theoretical tools discussed so far, such as Wick's theorem. We define the following quark and gluon propogators in terms of their contraction expressed in d dimensions:

$$
\begin{aligned}
\overbracket{c(x)\bar{c}}(y) &= \int \frac{d^d p}{(2\pi)^d} iS^{m_c}(p)\, e^{-ip\cdot(x-y)} \\
&\equiv \int \frac{d^d p}{(2\pi)^d} \frac{i(\not{p}+m_c)}{p^2 - m_c^2 + i\epsilon} e^{-ip\cdot(x-y)}
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
\overbracket{s(x)\bar{s}}(y) &= \int \frac{d^d p}{(2\pi)^d} iS^{m_s}(p)\, e^{-ip\cdot(x-y)} \\
&\equiv \int \frac{d^d p}{(2\pi)^d} \frac{i(\not{p}+m_s)}{p^2 - m_s^2 + i\epsilon} e^{-ip\cdot(x-y)}
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
\overbracket{B^a_\mu(x)B^b_\nu}(y) &= \int \frac{d^d p}{(2\pi)^d} i\delta^{ab} D_{\mu\nu}(p) e^{-ip\cdot(x-y)} \\
&\equiv -\int \frac{d^d p}{(2\pi)^d} \frac{i\delta^{ab}\, g_{\mu\nu}}{p^2} e^{-ip\cdot(x-y)}
\end{aligned}
\;.
\tag{2.14}
$$

Here, S and $D_{\mu\nu}$ are the momentum-space quark and gluon propogators, respectively. The following identity will be useful later in dealing with our non-perturbative contributions to the correlation function (the fermion propogator here is phrased in momentum-space):

$$
\frac{\partial}{\partial p_\mu} S(p) = -S(p)\,\gamma^\mu S(p)\,.
\tag{2.15}
$$

Using the definition of the gluon field strength tensor and (2.14), we can derive the following results:

$$
\begin{aligned}
\overbracket{\underline{G}^a_{\mu\rho}(x)\underline{G}^b_{\nu\sigma}}(y) &= \int \frac{d^d p}{(2\pi)^d} i\delta^{ab} H_{\mu\rho\nu\sigma}(p) e^{-ip\cdot(x-y)} \\
&\equiv -i\int \frac{d^d p}{(2\pi)^d} \frac{\delta^{ab}}{p^2} e^{-ip\cdot(x-y)} \left(g_{\rho\sigma}p_\mu p_\nu + g_{\mu\nu}p_\rho p_\sigma - g_{\rho\nu}p_\mu p_\sigma - g_{\mu\sigma}p_\rho p_\nu\right)
\end{aligned}
\tag{2.16}
$$

$$
\overbracket{\underline{G}^a_{\mu\rho}(x)B^b_\lambda}(y) = -\int \frac{d^d p}{(2\pi)^d} \frac{\delta^{ab}}{p^2} e^{-ip\cdot(y-x)} \left(g_{\lambda\sigma}p_\nu - g_{\lambda\nu}p_\sigma\right).
\tag{2.17}
$$

Note that definition $\underline{G}^a_{\mu\rho}(x) = \partial_\mu B^a_\rho(x) - \partial_\rho B^a_\mu(x)$ (the simplification introduced in (2.10)) separates the $\mathcal{O}(g_s^0)$ and $\mathcal{O}(g_s)$ terms in (2.3); in calculating the correlation function to leading order, we consider contributions at $\mathcal{O}(g_s^2)$. By defining $\underline{G}^a_{\mu\rho}(x)$, we can express our contractions in terms of this quantity which simplifies our calculation. The contraction of two full gluon field strength tensors introduces contributions at $\mathcal{O}\left(g_s^4\right)$ when considered in the context of our correlation function. However, we will later make use of the cross-terms produced in contracting two full gluon field strength tensors,

$$
\begin{aligned}
G^a_{\mu\rho}(x)G^b_{\nu\sigma}(y) = \quad & \underline{G}^a_{\mu\rho}(x)\underline{G}^b_{\nu\sigma}(y) + g_s f^{ade} B^d_\mu(x) B^e_\rho(x)\underline{G}^b_{\nu\sigma}(y) + g_s f^{ade} B^d_\mu(x) B^e_\rho(x)\underline{G}^b_{\nu\sigma}(y) \\
& + g_s f^{bfg}\underline{G}^a_{\mu\rho}(x) B^f_\nu(y) B^g_\sigma(y) + g_s f^{bfg}\underline{G}^a_{\mu\rho}(x) B^f_\nu(y) B^g_\sigma(y) + \mathcal{O}\left(g_s^2\right).
\end{aligned}
\tag{2.18}
$$

Another consideration that must be addressed is the colour structure of the calculations. Our current (2.7) contains the generators $t^a_{\alpha\beta} = \frac{\lambda^a_{\alpha\beta}}{2}$, where $\lambda^a_{\alpha\beta}$ are the Gell-Mann matrices (a is the gluon colour index, while $\alpha\,\beta$ are quark colour indices) which define the so-called antisymmetric structure constants, f^{abc}, as shown in (1.2). In evaluating the colour algebra in the calculation of our correlation function, we utilize the following identities relating generators and structure constants [16]:

$$
\begin{aligned}
t^a_{\alpha\beta} t^b_{\beta\alpha} = \mathrm{tr}\left[t^a t^b\right] &= \frac{\delta^{ab}}{2} \\
\left(t^a t^b\right)_{\alpha\beta} f^{abc} &= \frac{3i}{2} t^c_{\alpha\beta} \\
\left(t^a t^a t^b\right)_{\alpha\beta} &= \frac{4}{3} t^b_{\alpha\beta} \\
\left(t^a t^b t^a\right)_{\alpha\beta} &= -\frac{1}{6} t^b_{\alpha\beta} \\
\mathrm{tr}\left[t^a t^b t^c\right] &= \frac{1}{4}\left(d^{abc} + i f^{abc}\right).
\end{aligned}
\tag{2.19}
$$

Here, d^{abc} define the *symmetric* structure constants (in contrast to f^{abc}),

$$
\left\{t^a,\, t^b\right\} = \frac{1}{N}\delta^{ab} + d^{abc} t^c. \tag{2.20}
$$

2.2 Calculation Overview

As mentioned previously, the starting point for our calculations is to look at the correlation function formed using our hybrid current (2.11). The time-ordered products above may be expanded via Wick's theorem in such a way that we isolate the non-zero vacuum expectation values present in QCD. We expand the correlation function up to dimension six using (1.28), giving us an expression phrased in terms of local vacuum expectation values $\langle \overline{s}s \rangle$, $\langle \alpha GG \rangle$, $\langle \overline{s}\sigma \cdot Gs \rangle$, and $\langle g^3 G^3 \rangle$ (dimension three quark, dimension four gluon, dimension five mixed, and dimension six gluon condensates respectively). We begin from (2.11) and obtain

$$\begin{aligned}
\Pi_{\mu\nu}(q) = ig_s^2 \int & d^4x\, e^{iq\cdot x} \langle \Omega | T \overline{c}_i^{\alpha}(x) t^a_{\alpha\beta} \gamma^{\rho}_{ij} G^a_{\mu\rho}(x) s_j^{\beta}(x) \overline{s}_k^{\alpha}(0) t^b_{\alpha\beta} \gamma^{\sigma}_{kl} G^b_{\nu\sigma}(0) c_l^{\beta}(0) | \Omega \rangle \\
& + \frac{ig_s^2}{2} \int d^4x\, d^4y\, e^{iq\cdot x} \langle \Omega | T \overline{c}_i^{\alpha}(x) t_a^{\alpha\beta} \gamma^{\rho}_{ij} G^a_{\mu\rho}(x) s_j^{\beta}(x) \overline{s}_{\alpha}(y) \lambda^a_{\alpha\beta} \gamma^{\mu} s_{\beta}(y) B^a_{\mu}(y) \\
& \qquad \times \overline{s}_l^{\alpha}(0) t_b^{\alpha\beta} \gamma^{\sigma}_{lk} G^b_{\nu\sigma}(0) c_k^{\beta}(0) | \Omega \rangle \\
& + \frac{ig_s^2}{2} \int d^4x\, d^4y\, e^{iq\cdot x} \langle \Omega | T \overline{c}_i^{\alpha}(x) t_a^{\alpha\beta} \gamma^{\rho}_{ij} G^a_{\mu\rho}(x) s_j^{\beta}(x) \overline{c}_{\alpha}(y) \lambda^a_{\alpha\beta} \gamma^{\mu} c_{\beta}(y) B^a_{\mu}(y) \\
& \qquad \times \overline{s}_l^{\alpha}(0) t_b^{\alpha\beta} \gamma^{\sigma}_{lk} G^b_{\nu\sigma}(0) c_k^{\beta}(0) | \Omega \rangle \\
& - \frac{ig_s^2 f_{abc}}{2} \int d^4x\, d^4y\, e^{iq\cdot x} \langle \Omega | T \overline{c}_i^{\alpha}(x) t_a^{\alpha\beta} \gamma^{\rho}_{ij} G^a_{\mu\rho}(x) s_j^{\beta}(x) \left(\underline{G}^a_{\mu\nu}(x) B^{\mu}_b(x) B^{\nu}_c(x) \right) \\
& \qquad \times \overline{s}_l^{\alpha}(0) t_b^{\alpha\beta} \gamma^{\sigma}_{lk} G^b_{\nu\sigma}(0) c_k^{\beta}(0) | \Omega \rangle .
\end{aligned} \tag{2.21}$$

As mentioned previously in (1.12), the correlator $\Pi_{\mu\nu}(q)$ can be broken up into scalar and vector components; these components can be represented through the OPE as,

$$\begin{aligned}
\Pi_{V,S}\left(q^2\right) = C_{\text{pert}}\left(q^2\right) \mathbb{1} + C_{\overline{\text{s}}\text{s}}\left(q^2\right) \langle \overline{s}s \rangle + C_{\text{G}^2}\left(q^2\right) \langle \alpha GG \rangle \\
+ C_{\overline{\text{s}}\sigma\cdot\text{Gs}}\left(q^2\right) \langle \overline{s}\sigma \cdot Gs \rangle + C_{\text{G}^3}\left(q^2\right) \langle g^3 G^3 \rangle .
\end{aligned} \tag{2.22}$$

The coefficients C_{pert}, $C_{\overline{\text{s}}\text{s}}$, C_{G^2}, $C_{\overline{\text{s}}\sigma\cdot\text{Gs}}$, and C_{G^3} are determined through the Wick contractions of the time-ordered products above, as well as factors associated with the series expansion of the non-local VEVs. These contractions are summarized diagrammatically in Figures 2.1, 2.2, 2.3, and 2.4[1]. Our study of open-flavour systems proceeds in analogy to [25] where we have borrowed the diagram numbering scheme for convenience. The Roman numbering scheme of [25] appears in parentheses alongside the sequential Latin numbering adopted for the purposes of presentation in this thesis.

[1]All Feynman diagrams following created using Jaxodraw [24].

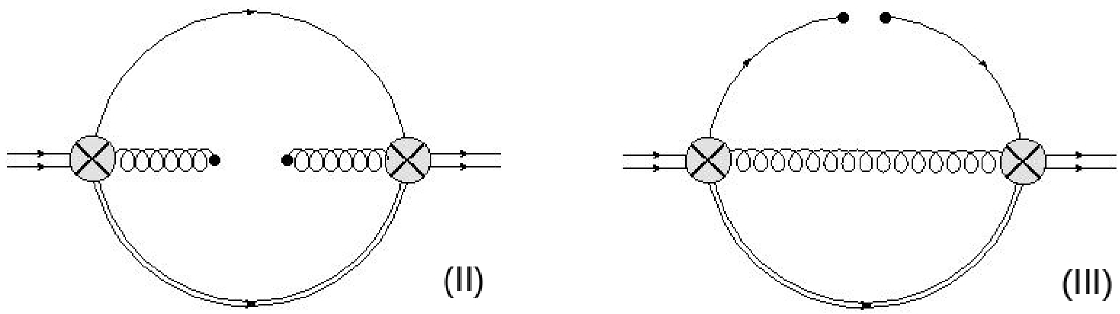

Figure 2.1: Feynman diagram depictions of dimension four gluon (II) and quark (III) condensates, parameterized by the condensates $\langle \alpha G^2 \rangle$ and $\langle \overline{q} q \rangle$ respectively.

2.3 Perturbative Contributions

Diagram 1 (I)

To begin, we examine the contributions to our correlation function from perturbation theory, which just corresponds to a single two-loop diagram at leading order, and a fully Wick-contracted term in the OPE

$$\begin{aligned}\Pi^{\text{pert}}_{\mu\nu}(q) &= i g_s^2 \int d^d x \, e^{iq\cdot x} \langle \Omega | \overline{c}_i^{\alpha}(x) t^a_{\alpha\beta} \gamma^{\rho}_{ij} G^a_{\mu\rho}(x) s_j^{\beta}(x) \overline{s}_k^{\beta'}(0) t^b_{\beta'\alpha'} \gamma^{\sigma}_{kl} G^b_{\nu\sigma}(0) c_l^{\alpha'}(0) | \Omega \rangle \\ &= -i^3 g_s^2 \, \text{tr}\left[t^a t^b\right] \int d^d x \, e^{iq\cdot x} S^{m_c}_{li}(-x) \gamma^{\rho}_{ij} S^{m_s}_{jk}(x) \gamma^{\sigma}_{kl} \\ &\qquad \times \left(-i \int \frac{d^d p}{(2\pi)^d} \frac{\delta^{ab}}{p^2} e^{-ip\cdot x} \left(g_{\rho\sigma} p_\mu p_\nu + g_{\mu\nu} p_\rho p_\sigma - g_{\rho\nu} p_\mu p_\sigma - g_{\mu\sigma} p_\rho p_\nu \right) \right). \end{aligned} \tag{2.23}$$

From here, it's convenient to move the calculation into momentum-space. To do this, we simply substitute the propagators phrased in terms of position space for their Fourier-transformed versions,

$$S(x) = \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x} S(k). \tag{2.24}$$

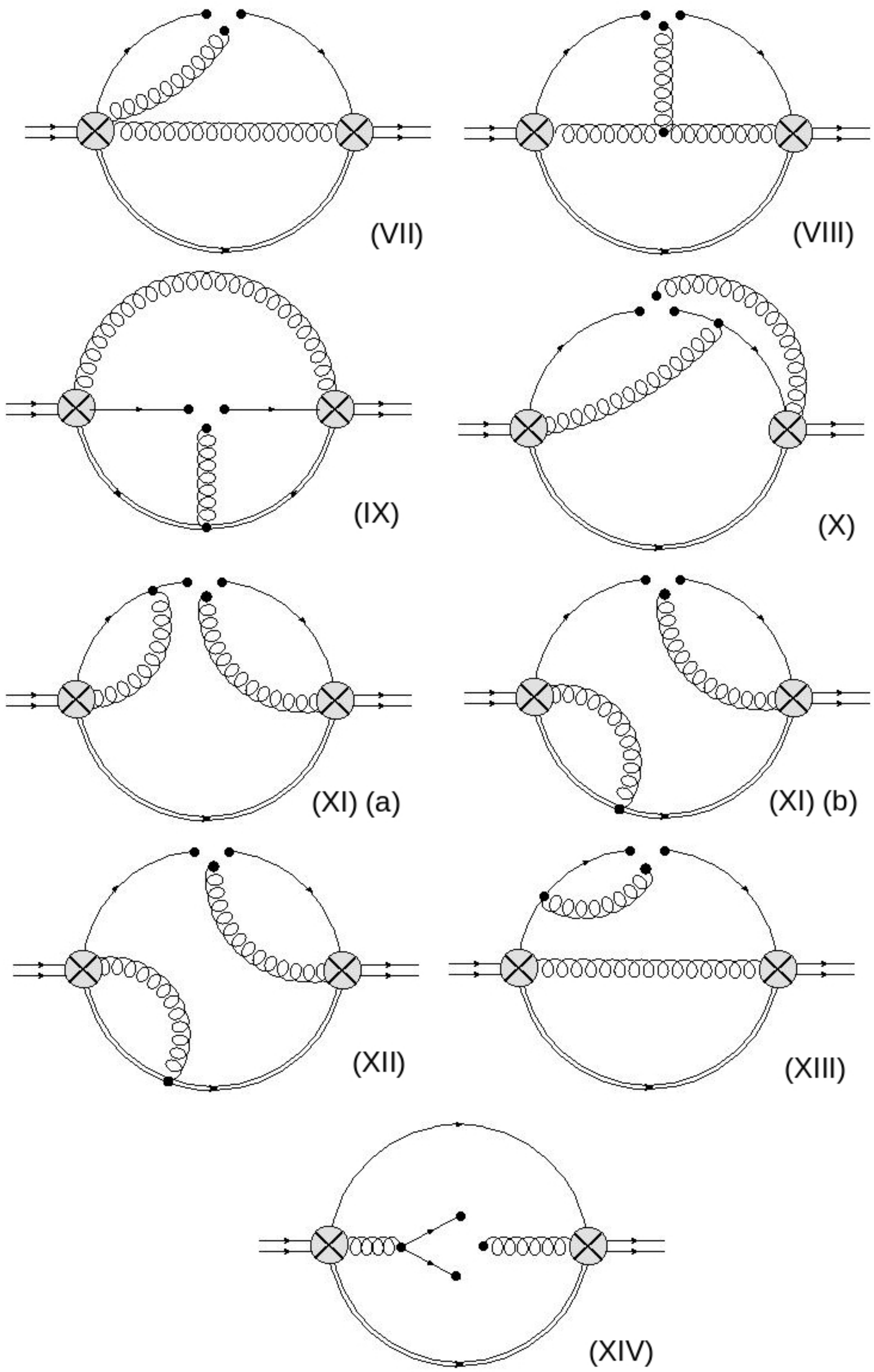

Figure 2.2: Feynman diagram depictions of the dimension five ("mixed") condensates, parameterized by the condensate $\langle \bar{q}\sigma \cdot Gq \rangle$.

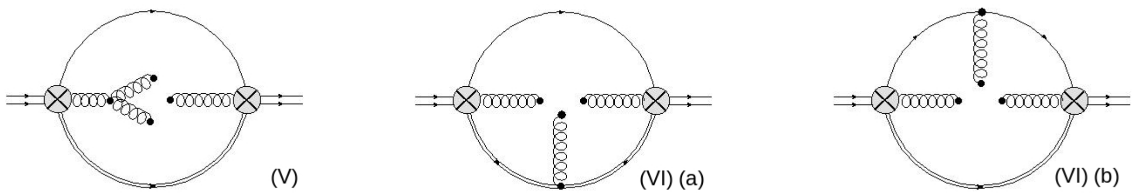

Figure 2.3: Feynman diagram depictions of the various contributions to the dimension six gluon condensate, parameterized by the condensate $\langle g^3 G^3 \rangle$.

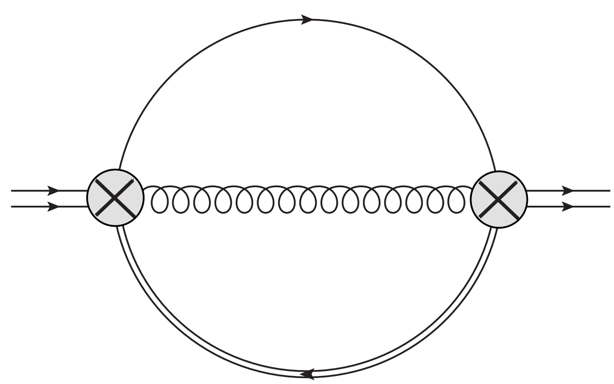

Figure 2.4: Feynman diagram representation of perturbation theory.

Replacing the charm and the strange quark propagators (S^{m_c} and S^{m_s}, respectively) with their momentum-space equivalents, we arrive at

$$\begin{aligned}\Pi^{\text{pert}}_{\mu\nu}(q) = g_s^2 \,\text{tr}\left[t^a t^a\right] \int \frac{d^d p\, d^d k}{(2\pi)^{2d}} \frac{1}{p^2} \text{tr}\left[S^{m_c}(k-q)\gamma^\rho S^{m_s}(p)\gamma^\sigma\right] \\ \times \left(g_{\rho\sigma} p_\mu p_\nu + g_{\mu\nu} p_\rho p_\sigma - g_{\rho\nu} p_\mu p_\sigma - g_{\mu\sigma} p_\rho p_\nu\right).\end{aligned} \tag{2.25}$$

By evaluating the colour and fermion traces utilizing results from (1.15) and (2.19), and contracting the ρ and σ indices, we arrive at

$$
\begin{aligned}
\Pi^{\text{pert}}_{\mu\nu}(q) &= 4g_s^2 \int \frac{d^d p\, d^d k}{(2\pi)^{2d}} \frac{\left((k_\lambda - q_\lambda)p_\kappa \mathrm{tr}\left[\gamma^\lambda\gamma^\rho\gamma^\kappa\gamma^\sigma\right] + m_c m_s \mathrm{tr}\left[\gamma^\rho\gamma^\sigma\right]\right)}{(k-p)^2((k-q)^2 - m_c^2)(p^2 - m_s^2)} \\
&\qquad\qquad \times \left(g_{\rho\sigma}p_\mu p_\nu + g_{\mu\nu}p_\rho p_\sigma - g_{\rho\nu}p_\mu p_\sigma - g_{\mu\sigma}p_\rho p_\nu\right) \\
&= 4g_s^2 \int \frac{d^d p\, d^d k}{(2\pi)^{2d}} \frac{1}{(k-p)^2((k-q)^2 - m_c^2)(p^2 - m_s^2)} \\
&\qquad \times \Big(\big(p^2(2k\cdot q + m_s m_c - p\cdot q) + k\cdot p\left(-2k\cdot q - 2m_s m_c + p^2\right) \\
&\qquad\qquad + k^2\left(k\cdot p + m_s m_c - 2p^2 + p\cdot q\right)\big)\, g_{\mu\nu} \\
&\qquad\quad + k_\mu\Big(k_\nu\left(-(d-2)k\cdot p + (d-2)m_s m_c + (d-4)p\cdot q + 2p^2\right) \\
&\qquad\qquad - p_\nu\left(-(d-2)k\cdot p + (d-2)m_s m_c - k\cdot q + k^2 + p^2 + (d-3)p\cdot q\right) \\
&\qquad\qquad + \left(k\cdot p - p^2\right) q_\nu\Big) \\
&\qquad\quad + p_\mu p_\nu((d-2)m_s m_c + (d-2)p\cdot q + 2k^2 + (2-d)k\cdot p - 2k\cdot q) \\
&\qquad\quad + p_\mu k_\nu((2-d)m_s m_c + (d-2)k\cdot p - p^2 - k^2 + k\cdot q + (3-d)p\cdot q) \\
&\qquad\quad + q_\mu k_\nu(k\cdot p - p^2) + (q_\mu p_\nu + p_\mu q_\nu)(p^2 - k\cdot p)\Big)
\end{aligned}
\tag{2.26}
$$

From here, we can extract the contribution to the vector ($J = 1$) state by applying (1.13). Thus,

$$\begin{aligned}
\Pi_v^{\text{pert}}\left(q^2\right) = 16 g_s^2 \int \frac{d^d p\, d^d k}{(2\pi)^d} & \frac{1}{(k-p)^2((k-q)^2 - m_c^2)(p^2 - m_s^2)} \\
& \times \frac{1}{(d-1)q^2} \Big(m_c m_s \left((d-2)(p\cdot q)^2 - (2d-3)q^2\left(k^2+p^2\right)\right) \\
& \qquad + 2k\cdot q\left(-p\cdot q\left((d-2)m_c m_s + (d-2)p\cdot q + k^2 + p^2\right)\right. \\
& \qquad\qquad \left. -(d-2)p^2 q^2\right) \\
& \qquad + k\cdot p\left(q^2\left(2(2d-3)m_c m_s + k^2 + p^2\right)\right. \\
& \qquad\qquad + 2(d-2)k\cdot q\left(p\cdot q + q^2\right) - (d-2)(k\cdot q)^2 \\
& \qquad\qquad \left. + 2(d-3)q^2 p\cdot q - (d-2)(p\cdot q)^2\right) \\
& \qquad + (k\cdot q)^2\left((d-2)m_c m_s + (d-2)p\cdot q + 2p^2\right) \\
& \qquad + 2(d-3)k^2 p^2 q^2 + p\cdot q\left(q^2\left((5-2d)k^2 + p^2\right)\right. \\
& \qquad \left. + (d-2)(p\cdot q)^2 + 2k^2 p\cdot q\right) - 2(d-2)q^2(k\cdot p)^2 \Big)
\end{aligned} \tag{2.27}$$

It is at this point where we expand our expression about $m_s = 0$ to first order. A series expansion in the strange quark mass allows us to reduce the number of dimensionful parameters in our integrands and simplify the evaluation of the integral:

$$
\begin{aligned}
\Pi_v^{\mathrm{pert}}\left(q^2\right) = \frac{16g_s^2}{(d-1)q^2} \Bigg(\int \frac{d^dp\, d^dk}{(2\pi)^d} & \frac{1}{(k-p)^2((k-q)^2-m_c^2)p^2} \\
\times \Big(2k\cdot q \Big(& -p\cdot q\left((d-2)p\cdot q + k^2 + p^2\right) \\
& - (d-2)p^2q^2 \Big) \\
+ k\cdot p \Big(& q^2\left(k^2+p^2\right) + 2(d-2)k\cdot q\left(p\cdot q + q^2\right) \\
& - (d-2)(k\cdot q)^2 + 2(d-3)q^2 p\cdot q \\
& - (d-2)(p\cdot q)^2 \Big) \\
+ (k\cdot q)^2 & \left((d-2)p\cdot q + 2p^2\right) + 2(d-3)k^2p^2q^2 \\
+ p\cdot q \Big(& q^2\left((5-2d)k^2 + p^2\right) + (d-2)(p\cdot q)^2 + 2k^2 p\cdot q \Big) \\
& - 2(d-2)q^2(k\cdot p)^2 \Big) \\
+ m_s \Bigg(\int \frac{d^dp\, d^dk}{(2\pi)^d} & \frac{1}{(k-p)^2((k-q)^2-m_c^2)p^2} \\
\times \Big(m_c & \left((d-2)(p\cdot q)^2 - (2d-3)q^2\left(k^2+p^2\right)\right) \\
& + k\cdot p\left(4dq^2m_c - 6q^2m_c\right) + m_c(k\cdot q)(p\cdot q)(4-2d) \\
& + m_c(k\cdot q)^2(d-2) \Big) \Bigg).
\end{aligned}
\tag{2.28}
$$

The $\mathcal{O}\left(m_s^0\right)$ contribution to this expression describes the perturbation theory contribution for the charm-light hybrid state. The addition of the $\mathcal{O}\left(m_s\right)$ correction allows us to look at the charm-strange hybrid case with the correct strange quark mass in mind. From here, we input our result into the TARCER package in order to apply Tarasov recursion relations, reducing the evaluation of our integrations to TARCER's master integrals,

$$
\begin{aligned}
\Pi_v^{\text{pert}}\left(q^2\right) = {} & \frac{2^{3-2d}\pi^{-d}g_s^2}{3(d-1)(3d-4)(3d-2)q^2} \\
& \times \Bigg(\left(\left(d^3-3d^2-4d+12\right)m_c^6+2\left(-39d^3+216d^2-382d+220\right)m_c^4q^2 \right.\\
& \quad \left. +\left(-39d^3+199d^2-324d+164\right)m_c^2q^4+4(d-2)^2(d-1)q^6\right)\mathbf{J}^d_{\{1,\,m_c\},\{1,0\},\{1,0\}} \\
& \quad -m_c^2\left(m_c^2-q^2\right)\left(\left(d^2-4\right)m_c^4-8\left(d^2-3d+2\right)q^4-(10-7d)^2m_c^2q^2\right) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbf{J}^d_{\{2,\,m_c\},\{1,0\},\{1,0\}}\Bigg) \\
& + m_s\Bigg(\frac{2^{3-2d}\pi^{-d}g_s^2m_c}{3(d-1)(3d-4)q^2} \\
& \quad \times \Bigg(\left(\left(7d^2-33d+36\right)m_c^2q^2+2\left(d^2-3d+2\right)q^4-(d-3)dm_c^4\right) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbf{J}^d_{\{1,\,m_c\},\{1,0\},\{1,0\}} \\
& \qquad\qquad + m_c^2\left(m_c^2-q^2\right)\left(dm_c^2+(8-5d)q^2\right)\mathbf{J}^d_{\{2,\,m_c\},\{1,0\},\{1,0\}}\Bigg)\Bigg).
\end{aligned}
\tag{2.29}
$$

Utilizing the results in [18] and [19], we can evaluate the **J**-type integrals appearing in our perturbative expression using (1.37). After evaluation of the integrals, we proceed to tackling the ϵ-expansion of the dimensional regularization. We take the same convention as [16], with $d = 4 + 2\epsilon$. From here, we expand about $\epsilon = 0$. To isolate any divergent behaviour about $\epsilon = 0$ in the hypergeometric functions appearing in (1.37), we use the definition of the hypergeometric functions in (1.38) as well as the involved Pochhammer symbols (1.39) to separate the divergences in ϵ before series expanding. Starting with an expression of the form

$$
f_1(\epsilon)\,{}_2F_1\left[\begin{matrix} a(\epsilon),\, b(\epsilon) \\ c(\epsilon) \end{matrix} \middle| z\right], \tag{2.30}
$$

where the functional dependence of ϵ has been denoted explicitly, inserting our expressions (1.38) and (1.39) we have

$$
f_1(\epsilon)\left(\frac{\Gamma\left(c(\epsilon)\right)}{\Gamma\left(a(\epsilon)\right)\Gamma\left(b(\epsilon)\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(a(\epsilon)+n\right)\Gamma\left(b(\epsilon)+n\right)}{\Gamma\left(c(\epsilon)+n\right)}\frac{z^n}{n!}\right). \tag{2.31}
$$

We selectively sum for explicit values of n until all divergences are removed from the sum as $\epsilon \to 0$:

$$f_1(\epsilon)\left(1 + \frac{\Gamma\left(c(\epsilon)\right)}{\Gamma\left(a(\epsilon)\right)\Gamma\left(b(\epsilon)\right)}\frac{\Gamma\left(a(\epsilon)+1\right)\Gamma\left(b(\epsilon)+1\right)}{\Gamma\left(c(\epsilon)+1\right)}z \right.$$
$$\left. + \ldots + \frac{\Gamma\left(c(\epsilon)\right)}{\Gamma\left(a(\epsilon)\right)\Gamma\left(b(\epsilon)\right)}\sum_{n=N}^{\infty}\frac{\Gamma\left(a(\epsilon)+N\right)\Gamma\left(b(\epsilon)+N\right)}{\Gamma\left(c(\epsilon)+N\right)}\frac{z^N}{N!}\right), \tag{2.32}$$

where N above is some integer in the sum where $\epsilon \to 0$ results in a finite contribution to the sum. We then expand both f_1 and the hypergeometric function about $\epsilon = 0$,

$$\left(\frac{a_1}{\epsilon^2} + \frac{b_1}{\epsilon} + c_1 + \ldots\right)\left(a_2 + b_2\epsilon + c_2\epsilon^2 + \ldots\right). \tag{2.33}$$

Note that the $\mathcal{O}\left(\frac{1}{\epsilon^m}\right)$ exhibited in f_1 determines the order in ϵ that we must expand our hypergeometric functions to in order to account for all finite contributions as $\epsilon \to 0$. After expanding about $\epsilon = 0$, we apply our $\overline{\text{MS}}$ renormalization prescription (1.33), and we are left with an expression of the form

$$
\begin{aligned}
\Pi_v^{\text{pert}} = {} & \frac{\alpha m_c^6(2z+1)}{32\pi^3\epsilon^2} - \frac{\alpha m_c^6\left(-120\log\left(\frac{m_c^2}{\nu^2}\right) - 240z\log\left(\frac{m_c^2}{\nu^2}\right) + 4z^3 - 45z^2 + 180z + 150\right)}{1920\pi^3\epsilon} \\
& + \frac{\alpha m_c^5}{345600\pi^3 z^2}\Bigg(- 360 m_c z^2\left(4z^3 - 45z^2 + 180z + 150\right)\log\left(\frac{m_c^2}{\nu^2}\right) \\
& \qquad + 7200 m_c(6z+3)z^2\log^2\left(\frac{m_c^2}{\nu^2}\right) \\
& \qquad + 7200 m_c(6z+3)z^2\mathrm{Li}_2(z) \\
& \qquad + 4212 m_c z^5 - 44865 m_c z^4 \\
& \qquad + 240 m_c z^3\left(45\pi^2 - 53 + 180\log(2)\right) \\
& \qquad + 360\left(157 + 15\pi^2\right) m_c z^2 - 360 m_c z \\
& \qquad + 360 m_c(1-z)(-z(1-z(-z(41-4z)-141))-1)\log(1-z)\Bigg) \\
& + m_s\Bigg(- \frac{\alpha m_c^5 z}{48\pi^3\epsilon^2} - \frac{\alpha m_c^5 z\left(24\log\left(\frac{m_c^2}{\nu^2}\right) + 3z - 22\right)}{576\pi^3\epsilon} \\
& \quad - \frac{\alpha m_c^5 z\mathrm{Li}_2(z)}{24\pi^3} + \frac{\alpha m_c^5(1-z)(1-z(5-z(3z+13)))\log(1-z)}{288\pi^3 z^2} \\
& \quad - \frac{\alpha m_c^5\left(-z\left(-z\left(-225z + 72\pi^2 + 116\right) - 132\right) - 24\right)}{6912\pi^3 z} \\
& \quad - \frac{\alpha m_c^5 z\log^2\left(\frac{m_c^2}{\nu^2}\right)}{24\pi^3} + \frac{\alpha m_c^5 z(22-3z)\log\left(\frac{m_c^2}{\nu^2}\right)}{288\pi^3}\Bigg),
\end{aligned}
\tag{2.34}
$$

where we have made the replacement $z = \frac{q^2}{m_c^2}$, and the polylogarithm $\mathrm{Li}_2(z)$ is defined by

$$
\mathrm{Li}_n(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^n}. \tag{2.35}
$$

Of concern in the correlation function is the z-dependence of the final ϵ-expanded expression. In the QCD sum rule analysis to extract the mass prediction, the Borel transform (1.42) eliminates all constant terms, and all polynomials in z. This eliminates all $\mathcal{O}\left(\epsilon^{-2}\right)$, $\mathcal{O}\left(\epsilon^{-1}\right)$, and $\mathcal{O}\left(\epsilon^{0}\right)$ terms with polynomial z-dependence, leaving us with the more complicated functions of z. It is from here that the QCD sum rule analysis begins, which is beyond the scope of this thesis.

2.4 Non-Perturbative Contributions

As we move forward from looking at the perturbative calculation to the non-perturbative contributions to the correlation function, we will pause to address the additional complications of evaluating the non-local vacuum expectation values arising from uncontracted field operators. The time-ordered product from (1.11) sheds light on the types of non-zero vacuum expectation values we will encounter:

$$\langle\Omega|T\overline{c}_i(x)G_{\mu\rho}(x)s_j(x)\overline{s}_k(0)G_{\nu\sigma}(0)c_l(0)|\Omega\rangle. \tag{2.36}$$

Keeping in mind the combinatoric nature of Wick's theorem described in (1.16), we can see that with appropriate combinations of contractions we will be left with[2]

$$\langle\overline{s}_k^{\beta'}(0)s_j^{\beta}(x)\rangle \tag{2.37}$$

$$\langle G^a_{\mu\rho}(x)G^b_{\nu\sigma}(0)\rangle, \tag{2.38}$$

which both satisfy the mass dimension ≤ 6 for the order of our particular calculation. Note that by leaving the strange quark and gluon field strength fields uncontracted, we could generate a dimension seven condensate; as this is beyond the order we are examining, we neglect these contributions. We note that a relationship between the heavy-quark condensate and the gluon condensates has previously been determined. Equation (27.52) in [15] describes an expansion of the quark condensate up to dimension six gluon condensates,

$$\langle\overline{q}q\rangle = -\frac{1}{12\pi m_q}\langle\alpha G^2\rangle - \frac{1}{1440\pi^2 m_q^3}\langle g^3G^3\rangle + \dots \tag{2.39}$$

where m_q represents the appropriate quark mass. As the information contained in the heavy quark condensate can be described through the gluon condensates (which we have already

[2] We have dropped the $N\{\}$ notation indicating normal-ordering, as well as the explicit bra- and ket- states to simplify notation.

included in our calculation), the heavy quark condensates are not included in OPE calculations to avoid the over-counting of states. For this reason, we do not consider the condensate associated with the charm quarks, only the strange quarks; the expansion above is performed for heavy quarks, and does not converge for the lighter case of the strange quark. Further, if we consider the higher-order radiative contributions as expressed in (2.11) and (2.22), we realize that more combinations arise; these non-local VEVs will eventually lead to expressions written in terms of the dimension five mixed and dimension six gluon condensates:

$$\langle \overline{s}_k^{\beta'}(z) B_\lambda^c(y) s_j^\beta(x) \rangle \tag{2.40}$$

$$\langle \overline{s}_k^{\beta'}(z) G_{\lambda\kappa}^c(y) s_j^\beta(x) \rangle \tag{2.41}$$

$$\langle G_{\mu\rho}^a(x) B_\lambda^c(y) G_{\nu\sigma}^b(z) \rangle \tag{2.42}$$

$$\langle G_{\mu\rho}^a(x) G_{\lambda\kappa}^c(y) G_{\nu\sigma}^b(z) \rangle . \tag{2.43}$$

In order to determine the form of these VEVs, we use the following ideas:

1. Locality: The non-zero VEVs are a property of the QCD vacuum, which is described through the metric $g_{\mu\nu}$ and the Dirac matrices.
2. The interacting vacuum $|\Omega\rangle$ is gauge and Lorentz invariant.
3. $G_{\mu\nu}^a(x)$ is antisymmetric under exchange of Lorentz indices.
4. The QCD equations of motion.

As an example, consider the non-local VEV in (2.38) As discussed in Chapter 1, the operator product expansion takes non-local operators such as (2.38) above, and expresses them in terms of a series of local operators. By taking a series expansion about $x = 0$, we find

$$\begin{aligned} \langle G_{\mu\rho}^a(x) G_{\nu\sigma}^b(0) \rangle \approx \langle G_{\mu\rho}^a(0) G_{\nu\sigma}^b(0) \rangle &+ x^\omega \langle \left(\partial_\omega G_{\mu\rho}^a(0)\right) G_{\nu\sigma}^b(0) \rangle \\ &+ \frac{x^{\omega_1} x^{\omega_2}}{2} \langle \left(\partial_{\omega_1} \partial_{\omega_2} G_{\mu\rho}^a(0)\right) G_{\nu\sigma}^b(0) \rangle + \ldots . \end{aligned} \tag{2.44}$$

Focusing on the $\mathcal{O}\left(x^0\right)$ term on the RHS of (2.44), we notice that the above combination of gluon field strength tensors are antisymmetric under the exchange of $\mu \leftrightarrow \nu$ or $\rho \leftrightarrow \sigma$, and for a colourless state (required by the gauge-invariance of the vacuum) we require $a = b$. From here we can extract the colour and Lorentz structure,

$$\langle G^a_{\mu\rho}(0) G^b_{\nu\sigma}(0)\rangle = C\,\delta^{ab}(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\rho\nu}). \tag{2.45}$$

To determine the coefficient C, we can contract both sides with $\delta^{ab}g^{\mu\nu}g^{\rho\sigma}$ to isolate C

$$\begin{aligned}\langle G^a_{\mu\sigma}(0) G^a_{\mu\sigma}(0)\rangle &= C\,\delta^{aa}(d^2 - d)\\ \Rightarrow C &= \frac{\langle GG\rangle}{8d(d-1)},\end{aligned} \tag{2.46}$$

where we have defined $\langle G^a_{\mu\sigma}(0) G^a_{\mu\sigma}(0)\rangle \equiv \langle GG\rangle$.

Considering the $\mathcal{O}\left(x\right)$ term in (2.44), we note that the Lorentz structure of the VEV is such that we are unable to build it solely from combinations of the metric (i.e., the VEV has an odd number of Lorentz indices). Since we are unable to construct the Lorentz structure, the contribution of the $\mathcal{O}\left(x\right)$ term must be zero.

Lastly, we will take a look at carefully deducing the form of the $\mathcal{O}\left(x^2\right)$ term in (2.44). According to [16], due to our gauge condition (2.6) we can exchange our conventional derivatives for an arrangement of gauge-covariant ones

$$x^{\omega_1}\partial_{\omega_1}G_{\mu\nu}(0) = x^{\omega_1}\left[D_{\omega_1}(0), G_{\mu\nu}(0)\right]. \tag{2.47}$$

Note the careful distinction between $G^a_{\mu\nu}$ and $G_{\mu\nu}$ (recall that we first introduced this in (2.4)). We can take our expansion (2.44) and include the appropriate colour terms to write our expansion in terms of $G_{\mu\nu}$,

$$\begin{aligned}-g_s^2(t^a t^b)\langle G^a_{\mu\rho}(x) G^b_{\nu\sigma}(0)\rangle = &-g_s^2(t^a t^b)\langle G^a_{\mu\rho}(0) G^b_{\nu\sigma}(0)\rangle + \ldots\\ &+\frac{1}{2}x^{\omega_1}x^{\omega_2}\langle\left[D_{\omega_1}(0), \left[D_{\omega_2}(0), G_{\mu\rho}(0)\right]\right] G_{\nu\sigma}(0)\rangle\end{aligned} \tag{2.48}$$

(note that the $\mathcal{O}\left(x\right)$ term in the expansion is zero as previously discussed). From this next-to-leading-order term, we can extract a contribution to the dimension six gluon condensate. The form of the VEV above has two symmetries that will help us contruct a general expression;

the expression is antisymmetric under the exchange of $\mu \leftrightarrow \rho$ and $\nu \leftrightarrow \sigma$. With this in mind, we construct a general form for the VEV [3]

$$\begin{aligned}\langle [D_{\omega_1}, [D_{\omega_2}, G_{\mu\rho}]]\, G_{\nu\sigma}\rangle = {} & A\, g_{\omega_1\omega_2}\left(g_{\mu\nu}g_{\rho\sigma} - g_{\rho\nu}g_{\mu\sigma}\right) \\ & + B\left[g_{\omega_2\nu}\left(g_{\omega_1\mu}g_{\rho\sigma} - g_{\omega_1\rho}g_{\mu\sigma}\right) - g_{\omega_2\sigma}\left(g_{\omega_1\mu}g_{\rho\nu} - g_{\omega_1\rho}g_{\mu\nu}\right)\right] \\ & + C\left[g_{\omega_1\nu}\left(g_{\omega_2\mu}g_{\rho\sigma} - g_{\omega_2\rho}g_{\mu\sigma}\right) - g_{\omega_1\sigma}\left(g_{\omega_2\mu}g_{\rho\nu} - g_{\omega_2\rho}g_{\mu\nu}\right)\right].\end{aligned} \tag{2.49}$$

Once again, by contracting with different combinations of metrics, we can solve for the coefficients A, B, and C. Consider the combinations: (1) $g^{\omega_2\mu}g^{\omega_1\nu}g^{\rho\sigma}$, (2) $g^{\omega_1\mu}g^{\omega_2\nu}g^{\rho\sigma}$, and (3) $g^{\omega_1\omega_2}g^{\mu\nu}g^{\rho\sigma}$. Applying (1) to the LHS of (2.49),

$$\begin{aligned}g^{\omega_2\mu}g^{\omega_1\nu}g^{\rho\sigma}\langle [D_{\omega_1}, [D_{\omega_2}, G_{\mu\rho}]]\, G_{\nu\sigma}\rangle &= \langle [D^{\nu}, [D^{\mu}, G_{\mu\rho}]]\, G_{\nu\rho}\rangle \\ &= \langle [D^{\nu}, J_{\rho}]\, G_{\nu\rho}\rangle\end{aligned} \tag{2.50}$$

where we have applied the equation of motion from [16]

$$[D^{\mu}(x), G_{\mu\nu}(x)] = -ig_s^2 t^a \sum_A \overline{q}^A(x) t^a \gamma_\nu q^A(x) \equiv J_\nu(x). \tag{2.51}$$

Here, we can utilize *translation invariance* to simplify our result in (2.50). Translation invariance arises from the idea that local VEVs are not dependent on their coordinate, i.e.

$$\begin{aligned}&\partial_\mu \langle A(0)B(0)\rangle = 0 \\ \Rightarrow\ &\langle (\partial_\mu A(0))\, B(0)\rangle = -\langle A(0)\,(\partial_\mu B(0))\rangle.\end{aligned} \tag{2.52}$$

This combined with (2.47) allows us to write (2.50) as

$$\begin{aligned}\langle [D^{\nu}, J_\rho]\, G_{\nu\rho}\rangle &= -\langle J_\rho\, [D^{\nu}, G_{\nu\rho}]\rangle \\ &= -\langle J_\rho J_\rho\rangle,\end{aligned} \tag{2.53}$$

[3] For the remainder of the calculation of this dimension six VEV, we will assume that covariant derivatives D_μ, currents J_μ, and gluon field strength operators $G_{\mu\nu}$ or $G^a_{\mu\nu}$ all act at the coordinate $x = 0$ unless explicitly specified.

where we have used the equation of motion (2.51). This term is of $\mathcal{O}\left(g_s^4\right)$ which is of a higher order than our calculation, thus it is neglected. Our final form of the contracted equation is

$$\begin{aligned} g^{\omega_2\mu}g^{\omega_1\nu}g^{\rho\sigma}\langle[D_{\omega_1},[D_{\omega_2},G_{\mu\rho}]]\,G_{\nu\sigma}\rangle = g^{\omega_2\mu}g^{\omega_1\nu}g^{\rho\sigma}\Big(&A\,g_{\omega_1\omega_2}\left(g_{\mu\nu}g_{\rho\sigma}-g_{\rho\nu}g_{\mu\sigma}\right)\\ &+B\left[g_{\omega_2\nu}\left(g_{\omega_1\mu}g_{\rho\sigma}-g_{\omega_1\rho}g_{\mu\sigma}\right)-g_{\omega_2\sigma}\left(g_{\omega_1\mu}g_{\rho\nu}-g_{\omega_1\rho}g_{\mu\nu}\right)\right]\\ &+C\left[g_{\omega_1\nu}\left(g_{\omega_2\mu}g_{\rho\sigma}-g_{\omega_2\rho}g_{\mu\sigma}\right)-g_{\omega_1\sigma}\left(g_{\omega_2\mu}g_{\rho\nu}-g_{\omega_2\rho}g_{\mu\nu}\right)\right]\Big)\\ \Rightarrow 0 = (d-1)d(A+B+(d-1)C).& \end{aligned} \tag{2.54}$$

Applying (2) to (2.49),

$$\begin{aligned} g^{\omega_1\mu}g^{\omega_2\nu}g^{\rho\sigma}\langle[D_{\omega_1},[D_{\omega_2},G_{\mu\rho}]]\,G_{\nu\sigma}\rangle &= \langle[D^\mu,[D^\nu,G_{\mu\rho}]]\,G^\rho_\nu\rangle\\ &= -\langle[D^\nu,[G_{\mu\rho},D^\mu]]\,G^\rho_\nu\rangle\\ &\quad -\langle[G_{\mu\rho},[D^\mu,D^\nu]]\,G^\rho_\nu\rangle, \end{aligned} \tag{2.55}$$

where we have used the Jacobi Identity

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0 \tag{2.56}$$

to rewrite the RHS. The first term on the RHS of (2.55) is an $\mathcal{O}\left(g_s^4\right)$ contribution as before in (2.53) which we neglect. For the second term, we apply the equation of motion (2.4) to get

$$g^{\omega_1\mu}g^{\omega_2\nu}g^{\rho\sigma}\langle[D_{\omega_1},[D_{\omega_2},G_{\mu\rho}]]\,G_{\nu\sigma}\rangle = \langle[G_{\mu\rho},G^{\mu\nu}]\,G^\rho_\nu\rangle \tag{2.57}$$

Using the definition for $G_{\mu\nu}$ in (2.4), we can write

$$\begin{aligned} \langle[G_{\mu\rho},G^{\mu\nu}]\,G^\rho_\nu\rangle &= -g_s^3\left[t^a,t^b\right]t^d\langle G^a_{\mu\rho}G^{\mu\nu b}G^{d\rho}_\nu\rangle\\ &= -ig_s^3 f^{abc}\mathrm{tr}\left[t^ct^d\right]\langle G^a_{\mu\rho}G^{\mu\nu b}G^{d\rho}_\nu\rangle\\ &= -\frac{ig_s^3 f^{abd}}{2}\langle G^a_{\mu\rho}G^{\mu\nu b}G^{d\rho}_\nu\rangle\\ &\equiv -\frac{ig_s^3}{2}\langle GGG\rangle, \end{aligned} \tag{2.58}$$

where we have grouped all the coloured terms into the definition of the VEV

$$\langle GGG \rangle \equiv f^{abd} \langle G^{a}_{\mu\rho} G^{\mu\nu b} G^{d\rho}_{\nu} \rangle. \tag{2.59}$$

Our final contracted equation is

$$\begin{aligned}
g^{\omega_1\mu} g^{\omega_2\nu} g^{\rho\sigma} \langle [D_{\omega_1}, [D_{\omega_2}, G_{\mu\rho}]]\, G_{\nu\sigma} \rangle = g^{\omega_1\mu} g^{\omega_2\nu} g^{\rho\sigma} \Big(& A\, g_{\omega_1\omega_2} \left(g_{\mu\nu} g_{\rho\sigma} - g_{\rho\nu} g_{\mu\sigma} \right) \\
& + B \left[g_{\omega_2\nu} \left(g_{\omega_1\mu} g_{\rho\sigma} - g_{\omega_1\rho} g_{\mu\sigma} \right) - g_{\omega_2\sigma} \left(g_{\omega_1\mu} g_{\rho\nu} - g_{\omega_1\rho} g_{\mu\nu} \right) \right] \\
& + C \left[g_{\omega_1\nu} \left(g_{\omega_2\mu} g_{\rho\sigma} - g_{\omega_2\rho} g_{\mu\sigma} \right) - g_{\omega_1\sigma} \left(g_{\omega_2\mu} g_{\rho\nu} - g_{\omega_2\rho} g_{\mu\nu} \right) \right] \Big) \\
\Rightarrow -\frac{i g_s^3}{2} \langle GGG \rangle = (-1+d) d (A + B(-1+d) + C).
\end{aligned} \tag{2.60}$$

In much the same way, we can show that the application of projector (3), $g^{\omega_1\omega_2} g^{\mu\nu} g^{\rho\sigma}$, gives us

$$-\, i g_s^3 \langle GGG \rangle = (-1+d) d (2B + Ad + 2C). \tag{2.61}$$

Taking equations (2.54), (2.60), and (2.61), we determine

$$A = -\frac{i g_s^3 \langle GGG \rangle}{d(d-2)(d+2)} \tag{2.62}$$

$$B = -\frac{i g_s^3 \langle GGG \rangle}{2d(d-2)(d+2)} \tag{2.63}$$

$$C = \frac{3 i g_s^3 \langle GGG \rangle}{2d(d-1)(d-2)(d+2)}. \tag{2.64}$$

Thus,

$$
\begin{aligned}
\langle [D_{\omega_1}, [D_{\omega_2}, G_{\mu\rho}]]\, G_{\nu\sigma}\rangle = & -\frac{ig_s^3\langle GGG\rangle}{d(d-2)(d+2)}\, g_{\omega_1\omega_2}\,(g_{\mu\nu}g_{\rho\sigma} - g_{\rho\nu}g_{\mu\sigma}) \\
& - \frac{ig_s^3\langle GGG\rangle}{2d(d-2)(d+2)} \\
& \quad \times \left[g_{\omega_2\nu}\,(g_{\omega_1\mu}g_{\rho\sigma} - g_{\omega_1\rho}g_{\mu\sigma}) - g_{\omega_2\sigma}\,(g_{\omega_1\mu}g_{\rho\nu} - g_{\omega_1\rho}g_{\mu\nu})\right] \\
& + \frac{3ig_s^3\langle GGG\rangle}{2d(d-1)(d-2)(d+2)} \\
& \quad \times \left[g_{\omega_1\nu}\,(g_{\omega_2\mu}g_{\rho\sigma} - g_{\omega_2\rho}g_{\mu\sigma}) - g_{\omega_1\sigma}\,(g_{\omega_2\mu}g_{\rho\nu} - g_{\omega_2\rho}g_{\mu\nu})\right].
\end{aligned}
\tag{2.65}
$$

To conclude, the necessary relationships between the non-local VEVs and the condensates are as follows [26, 27]:

$$
\langle G^a_{\mu\rho}(x)G^b_{\nu\sigma}(0)\rangle = \frac{\langle GG\rangle}{8d(d-1)}\,\delta^{ab}(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\rho\nu}) \tag{2.66}
$$

$$
\langle \overline{s}^{\beta'}_k(x)s^{\beta}_j(0)\rangle = \frac{\langle \overline{s}s\rangle}{12}\delta^{\beta\beta'}\delta_{jk} \tag{2.67}
$$

$$
t^c_{\beta'\beta}\langle \overline{s}^{\beta'}_k(x)B^c_\lambda(y)s^{\beta}_j(0)\rangle = \frac{\langle \overline{s}\sigma\cdot Gs\rangle}{8id(d-1)g_s}y_\omega(\sigma^{\omega\lambda})_{jk} \tag{2.68}
$$

$$
t^c_{\beta'\beta}\langle \overline{s}^{\beta'}_k(x)G^c_{\lambda\kappa}(y)s^{\beta}_j(0)\rangle = \frac{\langle \overline{s}\sigma\cdot Gs\rangle}{4id(d-1)g_s}(\sigma^{\lambda\kappa})_{jk} \tag{2.69}
$$

$$
\begin{aligned}
g_s^3 f^{abc}\langle G^a_{\eta\tau}(x)G^c_{\sigma\alpha}(y)G^b_{\beta\lambda}(0)\rangle = \frac{\langle g^3G^3\rangle}{d(d^2-4)} & \left[(g_{\tau\sigma}g_{\alpha\beta}g_{\lambda\eta} - g_{\eta\sigma}g_{\alpha\beta}g_{\lambda\tau})\right. \\
& - (g_{\tau\alpha}g_{\sigma\beta}g_{\lambda\eta} - g_{\eta\alpha}g_{\sigma\beta}g_{\lambda\tau}) \\
& - (g_{\tau\sigma}g_{\alpha\lambda}g_{\beta\eta} - g_{\eta\sigma}g_{\alpha\lambda}g_{\beta\tau}) \\
& \left. + (g_{\tau\alpha}g_{\sigma\lambda}g_{\beta\eta} - g_{\eta\alpha}g_{\sigma\lambda}g_{\beta\tau})\right].
\end{aligned}
\tag{2.70}
$$

$$
\begin{aligned}
\langle [D_{\omega_1}, [D_{\omega_2}, G_{\mu\rho}]]\, G_{\nu\sigma}\rangle = &- \frac{i\langle g^3 G^3\rangle}{d(d-2)(d+2)}\, g_{\omega_1\omega_2}\left(g_{\mu\nu}g_{\rho\sigma} - g_{\rho\nu}g_{\mu\sigma}\right) \\
&- \frac{i\langle g^3 G^3\rangle}{2d(d-2)(d+2)} \\
&\quad \times \left[g_{\omega_2\nu}\left(g_{\omega_1\mu}g_{\rho\sigma} - g_{\omega_1\rho}g_{\mu\sigma}\right) - g_{\omega_2\sigma}\left(g_{\omega_1\mu}g_{\rho\nu} - g_{\omega_1\rho}g_{\mu\nu}\right)\right] \\
&+ \frac{3i\langle g^3 G^3\rangle}{2d(d-1)(d-2)(d+2)} \\
&\quad \times \left[g_{\omega_1\nu}\left(g_{\omega_2\mu}g_{\rho\sigma} - g_{\omega_2\rho}g_{\mu\sigma}\right) - g_{\omega_1\sigma}\left(g_{\omega_2\mu}g_{\rho\nu} - g_{\omega_2\rho}g_{\mu\nu}\right)\right].
\end{aligned}
\tag{2.71}
$$

2.4.1 Dimension Four Condensate Contributions

The calculation for the dimension four gluon condensate and the quark condensate proceed at leading order in g_s, without the introduction of any terms from the interaction Lagrangian.

Diagram 2 (II)

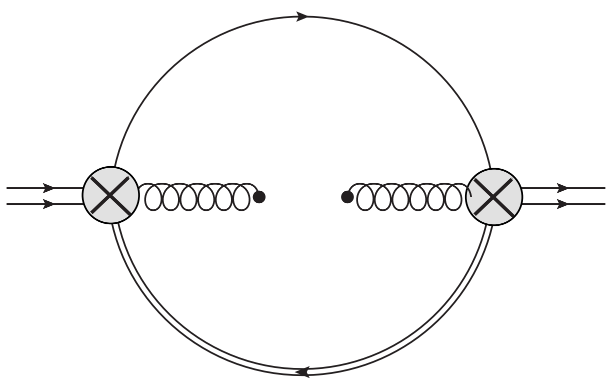

Figure 2.5: Dimension four condensate contribution, diagram 2 (II).

Starting from the full time-ordered product in equation (2.11), and proceeding similarly as in the previous perturbative calculation, we can obtain the dimension four gluon condensate result by leaving the gluon field strength tensors uncontracted in a normal-ordered product as

$$
\Pi^{\mathrm{GG}}_{\mu\nu}(q) = i g_s^2 \int d^d x\, e^{iq\cdot x} \langle \Omega | \overline{c}^{\alpha}_{i}(x) t^{a}_{\alpha\beta}\gamma^{\rho}_{ij}\underline{G}^{a}_{\mu\rho}(x) s^{\beta}_{j}(x)\overline{s}^{\beta'}_{k}(0) t^{b}_{\beta'\alpha'}\gamma^{\sigma}_{kl}\underline{G}^{b}_{\nu\sigma}(0) c^{\alpha'}_{l}(0)|\Omega\rangle, \tag{2.72}
$$

leading to the coordinate-space result

$$\Pi^{GG}_{\mu\nu} = \frac{ig_s^2}{2}\mathrm{tr}\left[t^a t^b\right]\int d^dx\, e^{iq\cdot x}\mathrm{tr}\left[S^{m_c}\left(x\right)\gamma^\sigma S^{m_s}\left(-x\right)\gamma^\rho\right]\langle G^a_{\mu\rho}(x)G^b_{\nu\sigma}(0)\rangle. \tag{2.73}$$

Moving into momentum space, we project out the vector component as demonstrated in equation (1.13) and apply the Tarasov recursion relations via TARCER to obtain

$$\begin{aligned}\Pi^{GG}_v(q^2) = -\frac{i2^{1-d}\pi^{-\frac{d}{2}}g_s^2\langle GG\rangle}{d(d-1)^2q^2}\Big(&\left((-(d-4)d-5)m_c^2q^2+(d-2)^2q^4+m_c^4\right)\mathbf{B}^d_{\{1,0\}}\\ &-\left((d-2)^2q^2+m_c^2\right)\mathbf{A}^d_{\{1,M\}}\Big).\end{aligned} \tag{2.74}$$

Evaluating the **A**- and **B**-type integrals, we ϵ-expand our result, apply our $\overline{\mathrm{MS}}$ renormalization scheme, and obtain our final result

$$\begin{aligned}\Pi^{GG}_v(q^2) =&\frac{m_c^2\left(9-4z\right)\langle\alpha GG\rangle}{144\epsilon}\\ &+\frac{m_c^2\langle\alpha GG\rangle}{864z^2}\Bigg(-6(4z-9)z^2\log\left(\frac{m_c^2}{\nu^2}\right)\\ &\qquad+44z^3-87z^2+6z-6(z-1)^2(4z-1)\log(1-z)\Bigg).\end{aligned} \tag{2.75}$$

Diagram 3 (III)

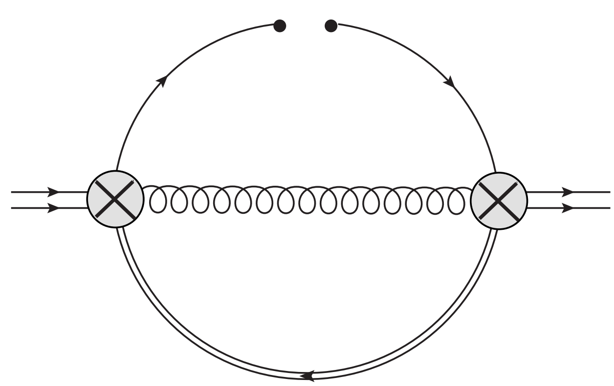

Figure 2.6: Dimension four condensate contribution, diagram 3 (III).

Similarly as with the dimension four gluon condensate, the dimension four quark condensate is formed by leaving the light quark fields uncontracted

$$\Pi^{\overline{q}q}_{\mu\nu}(q) = ig_s^2 \int d^d x \, e^{iq\cdot x} \langle \Omega | \overline{c}^{\alpha}_i(x) t^a_{\alpha\beta} \gamma^{\rho}_{ij} G^a_{\mu\rho}(x) s^{\beta}_j(x) \overline{s}^{\beta'}_k(0) t^b_{\beta'\alpha'} \gamma^{\sigma}_{kl} G^b_{\nu\sigma}(0) c^{\alpha'}_l(0) | \Omega \rangle. \tag{2.76}$$

As we're concerned with only $\mathcal{O}\left(g_s^2\right)$ contributions at the dimension four condensate level, we consider the contraction $G^a_{\mu\rho}(x) G^b_{\nu\sigma}(0) \approx \underline{G}^a_{\mu\rho}(x) \underline{G}^b_{\nu\sigma}(0)$. We will consider the implications of higher-order contributions in (2.18) later on when discussing the mixed condensate diagrams. The result of applying Wick's theorem (after using (2.67)) is

$$\Pi^{\overline{q}q}_{\mu\nu}(q) = -\frac{d g_s^2 \langle \overline{s}s \rangle}{24} \int d^d x \, e^{iq\cdot x} \mathrm{tr}\left[\gamma^{\sigma} S^{m_c}(x) \gamma^{\rho}\right] H_{\mu\rho\nu\sigma}(x). \tag{2.77}$$

Switching to momentum space gives us

$$\Pi^{\overline{q}q}_{\mu\nu}(q) = \frac{d g_s^2 \langle \overline{s}s \rangle}{24} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} \mathrm{tr}\left[\gamma^{\sigma} S^{m_c}(q-k) \gamma^{\rho}\right] \left(g_{\rho\sigma} k_\mu k_\nu + g_{\mu\nu} k_\rho k_\sigma - g_{\mu\sigma} k_\rho k_\nu - g_{\rho\nu} k_\mu k_\sigma\right). \tag{2.78}$$

After projecting out the vector component, and applying TARCER, we obtain

$$\begin{aligned} \Pi^{\overline{q}q}_{v}\left(q^2\right) = \frac{i g_s^2 m_c \langle \overline{s}s \rangle}{3(4\pi)^{\frac{d}{2}} (d-1) q^2} \Bigg(& \left((2-d) m_c^2 + (6-5d) q^2\right) \mathbf{A}^d_{\{1,\, m_c\}} \\ & + (d-2)\left(m_c^2 - q^2\right)^2 \mathbf{B}^d_{\{1, m_c\}\{1,0\}} \Bigg). \end{aligned} \tag{2.79}$$

Our final expression, applying the ϵ-expansion and renormalization scheme, is

$$\begin{aligned} \Pi^{\overline{q}q}_{v}\left(q^2\right) = & \frac{3\alpha_s m_c^3 \langle \overline{s}s \rangle (z-9)}{54\epsilon} \\ & + \frac{\alpha_s m_c^3 \langle \overline{s}s \rangle \left(3(z-9) z^2 \log\left(\frac{M^2}{\nu^2}\right) - 5z^3 + 30z^2 - 3z + 3(z-1)^3 \log(1-z)\right)}{54 z^2}. \end{aligned} \tag{2.80}$$

2.4.2 Dimension Five ("Mixed") Condensate Contributions

To address the dimension five condensate contributions, we must consider the contributions from the interaction Lagrangian (2.9) introduced previously. The dimension five condensate contributions are characterized by the VEV $\langle \bar{s}\sigma \cdot G s\rangle$ defined in (1.24).

Diagram 4 (VII)

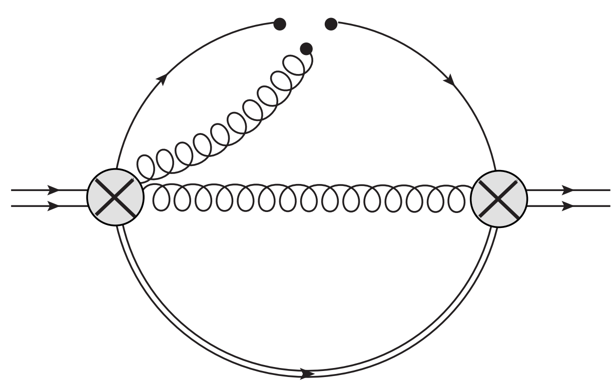

Figure 2.7: Mixed condensate contribution, diagram 4 (VII).

The contribution to the dimension five condensate arises from consideration of the full gluon field strength tensor (2.18) discussed previously. Up until now, we have ignored the $\mathcal{O}(g_s)$ term above, as it would only contribute to higher-order corrections in our overall calculation. Here, we consider these higher-order terms, which end up generating an additional gluon line from our current insertions (depicted in Figure 2.7), thus contributing to the dimension five mixed condensates. Utilizing definitions (2.16) and (2.17), we can develop a complete expression for the contraction of two gluon field strength tensors in momentum-space:

$$
\begin{aligned}
G^a_{\mu\rho}(x)G^b_{\nu\sigma}(z) = &-i\int \frac{d^dk}{(2\pi)^d}\frac{\delta^{ab}}{k^2}e^{-ik\cdot(x-z)}\left(g_{\rho\sigma}k_\mu k_\nu + g_{\mu\nu}k_\rho k_\sigma - g_{\rho\nu}k_\mu k_\sigma - g_{\mu\sigma}k_\rho k_\nu\right)\\
&+ g_s f^{ade}\Bigg(B^d_\mu(x)\int \frac{d^dk}{(2\pi)^d}\frac{\delta^{be}}{k^2}e^{-ik\cdot(x-z)}\left(g_{\rho\nu}k_\sigma - g_{\rho\sigma}k_\nu\right)\\
&\qquad + B^e_\rho(x)\int \frac{d^dk}{(2\pi)^d}\frac{\delta^{bd}}{k^2}e^{-ik\cdot(x-z)}\left(g_{\mu\nu}k_\sigma - g_{\mu\sigma}k_\nu\right)\Bigg)\\
&- g_s f^{bfg}\Bigg(B^f_\nu(z)\int \frac{d^dk}{(2\pi)^d}\frac{\delta^{ag}}{k^2}e^{-ik\cdot(z-x)}\left(g_{\mu\sigma}k_\rho - g_{\rho\sigma}k_\mu\right)\\
&\qquad + B^g_\sigma(z)\int \frac{d^dk}{(2\pi)^d}\frac{\delta^{af}}{k^2}e^{-ik\cdot(z-x)}\left(g_{\mu\nu}k_\rho - g_{\rho\nu}k_\mu\right)\Bigg).
\end{aligned}
\tag{2.81}
$$

From here, we can revisit our calculation of the dimension four quark condensate depicted in Figure 2.6, and use the complete form of the gluon field strength propogator to examine the contributions to the mixed condensate

$$
\begin{aligned}
\Pi^{\bar{q}q,\,\bar{q}\sigma Gq}_{\mu\nu}(q) &= ig_s^2\int d^dx\, e^{iq\cdot x}\langle\Omega|\bar{c}^\alpha_i(x)t^a_{\alpha\beta}\gamma^\rho_{ij}G^a_{\mu\rho}(x)s^\beta_j(x)\bar{s}^{\beta'}_k(0)t^b_{\beta'\alpha'}\gamma^\sigma_{kl}G^b_{\nu\sigma}(0)c^{\alpha'}_l(0)|\Omega\rangle\\
&= g_s^2\int d^dx\, e^{iq\cdot x}\, t^a_{\alpha\beta}t^b_{\beta'\alpha'}\,\mathrm{tr}\left[\gamma^\sigma S^{m_c}(x)\gamma^\rho\right]\\
&\quad\times\Bigg(i\delta^{ab}\langle\bar{s}^{\beta'}_k(0)s^\beta_j(x)\rangle\int\frac{d^dk}{(2\pi)^d}\frac{e^{-ik\cdot x}}{k^2}\left(g_{\rho\sigma}k_\mu k_\nu + g_{\mu\nu}k_\rho k_\sigma - g_{\mu\sigma}k_\rho k_\nu - g_{\rho\nu}k_\mu k_\sigma\right)\\
&\qquad + g_s f^{ade}\Bigg(\langle\bar{s}^{\beta'}_k(0)B^d_\mu(x)s^\beta_j(x)\rangle\int\frac{d^dk}{(2\pi)^d}\frac{\delta^{be}}{k^2}e^{-ik\cdot x}\left(g_{\rho\nu}k_\sigma - g_{\rho\sigma}k_\nu\right)\\
&\qquad + \langle\bar{s}^{\beta'}_k(0)B^e_\rho(x)s^\beta_j(x)\rangle\int\frac{d^dk}{(2\pi)^d}\frac{\delta^{bd}}{k^2}e^{-ik\cdot x}\left(g_{\mu\nu}k_\sigma - g_{\mu\sigma}k_\nu\right)\Bigg)\\
&\qquad - g_s f^{bfg}\Bigg(\langle\bar{s}^{\beta'}_k(0)B^f_\nu(0)s^\beta_j(x)\rangle\int\frac{d^dk}{(2\pi)^d}\frac{\delta^{ag}}{k^2}e^{-ik\cdot(-x)}\left(g_{\mu\sigma}k_\rho - g_{\rho\sigma}k_\mu\right)\\
&\qquad + \langle\bar{s}^{\beta'}_k(0)B^g_\sigma(0)s^\beta_j(x)\rangle\int\frac{d^dk}{(2\pi)^d}\frac{\delta^{af}}{k^2}e^{-ik\cdot(-x)}\left(g_{\mu\nu}k_\rho - g_{\rho\nu}k_\mu\right)\Bigg)\Bigg).
\end{aligned}
\tag{2.82}
$$

We identify the first term in the contribution above as the previously considered dimension four quark condensate (2.78); this piece is dropped, and we are left with the contributions to the dimension five mixed condensate. Using (2.68) (which eliminates any terms with a gluon field B evaluated at the origin), our expression simplifies down to

$$
\begin{aligned}
\Pi^{\bar{q}\sigma Gq}_{\mu\nu}(q) = {} & \frac{g_s^3}{2}\int d^d x\, e^{iq\cdot x}\, \left(t^b t^a\right)_{\beta'\beta}\, \gamma^\sigma_{kl} S^{m_c}_{li}(x)\gamma^\rho_{ij} \\
& \times x^\omega f^{ade}\Bigg(\langle \bar{s}^{\beta'}_k(0) G^d_{\omega\mu}(x) s^\beta_j(x)\rangle \int \frac{d^d k}{(2\pi)^d}\frac{\delta^{be}}{k^2} e^{-ik\cdot x}\left(g_{\rho\nu}k_\sigma - g_{\rho\sigma}k_\nu\right) \\
& \qquad + \langle \bar{s}^{\beta'}_k(0) G^e_{\omega\rho}(x) s^\beta_j(x)\rangle \int \frac{d^d k}{(2\pi)^d}\frac{\delta^{bd}}{k^2} e^{-ik\cdot x}\left(g_{\mu\nu}k_\sigma - g_{\mu\sigma}k_\nu\right)\Bigg) \\
= {} & -\frac{3ig_s^2\langle\bar{s}\sigma\cdot Gs\rangle}{16d(d-1)}\int \frac{d^d k}{(2\pi)^d}\frac{1}{k^2} \\
& \times \Bigg(\mathrm{tr}\left[\gamma^\sigma S^{m_c}(q-k)\gamma^\omega S^{m_c}(q-k)\gamma^\rho\sigma_{\omega\mu}\right]\left(g_{\rho\nu}k_\sigma - g_{\rho\sigma}k_\nu\right) \\
& \qquad - \mathrm{tr}\left[\gamma^\sigma S^{m_c}(q-k)\gamma^\omega S^{m_c}(q-k)\gamma^\rho\sigma_{\omega\rho}\right]\left(g_{\mu\nu}k_\sigma - g_{\mu\sigma}k_\nu\right)\Bigg).
\end{aligned}
\tag{2.83}
$$

Expanding, extracting the vector component, and applying TARCER, we obtain

$$
\Pi^{\bar{q}\sigma Gq}_{v}\left(q^2\right) = -\frac{3(d-2)g_s^2 m_c\langle\bar{s}\sigma\cdot Gs\rangle}{2^{d+3}\pi^{\frac{d}{2}}(d-1)d\,q\cdot q}\left(\left((2d-5)q\cdot q + m_c^2\right)\mathbf{B}^d_{\{1,m_c\},\{1,0\}} - \mathbf{A}^d_{\{1,m_c\}}\right), \tag{2.84}
$$

which evaluates to the ϵ-expanded expression

$$
\begin{aligned}
\Pi^{\bar{q}\sigma Gq}_{v}\left(q^2\right) = {} & \frac{3ig_s^2 m_c\langle\bar{s}\sigma\cdot Gs\rangle}{256\pi^2\epsilon} \\
& + \frac{ig_s^2 m_c\langle\bar{s}\sigma\cdot Gs\rangle\left(6z^2\log\left(\frac{m_c^2}{\nu^2}\right) + \left(6z^2-4z-2\right)\log(1-z) - z(5z+2)\right)}{512\pi^2 z^2}.
\end{aligned}
\tag{2.85}
$$

Diagram 5 (VIII)

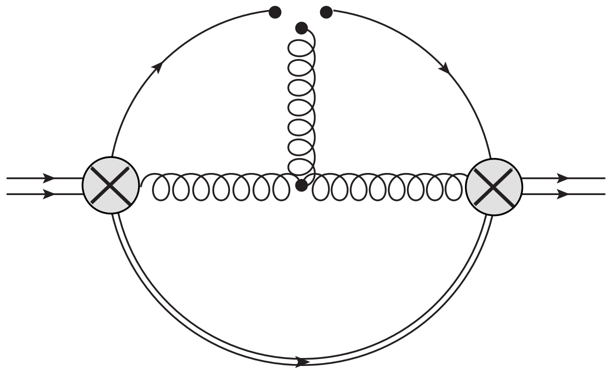

Figure 2.8: Mixed condensate contribution, diagram 5 (VIII).

The three gluon vertex from the interaction Langrangian (2.9) results in this mixed condensate contribution:

$$
\begin{aligned}
\Pi_{\mu\nu}^{\bar{q}\sigma G q}(q) = -\frac{i^2 g_s^3}{2} f^{cde} t^a_{\alpha\beta} t^b_{\beta'\alpha'} \int d^d x\, d^d y\, e^{iq\cdot x} \gamma^\rho_{ij} \gamma^\sigma_{kl} \\
\times \Big(\langle\Omega|\bar{c}_i^\alpha(x) G^a_{\mu\rho}(x) s_j^\beta(x) \underline{G}^c_{\epsilon\tau}(y) B^\epsilon_d(y) B^\tau_e(y) \bar{s}_k^{\beta'}(0) G^b_{\nu\sigma}(0) c_l^{\alpha'}(0)|\Omega\rangle \\
+ \langle\Omega|\bar{c}_i^\alpha(x) G^a_{\mu\rho}(x) s_j^\beta(x) \underline{G}^c_{\epsilon\tau}(y) B^\epsilon_d(y) B^\tau_e(y) \bar{s}_k^{\beta'}(0) G^b_{\nu\sigma}(0) c_l^{\alpha'}(0)|\Omega\rangle \\
+ \langle\Omega|\bar{c}_i^\alpha(x) G^a_{\mu\rho}(x) s_j^\beta(x) \underline{G}^c_{\epsilon\tau}(y) B^\epsilon_d(y) B^\tau_e(y) \bar{s}_k^{\beta'}(0) G^b_{\nu\sigma}(0) c_l^{\alpha'}(0)|\Omega\rangle \\
+ \langle\Omega|\bar{c}_i^\alpha(x) G^a_{\mu\rho}(x) s_j^\beta(x) \underline{G}^c_{\epsilon\tau}(y) B^\epsilon_d(y) B^\tau_e(y) \bar{s}_k^{\beta'}(0) G^b_{\nu\sigma}(0) c_l^{\alpha'}(0)|\Omega\rangle \\
+ \langle\Omega|\bar{c}_i^\alpha(x) G^a_{\mu\rho}(x) s_j^\beta(x) \underline{G}^c_{\epsilon\tau}(y) B^\epsilon_d(y) B^\tau_e(y) \bar{s}_k^{\beta'}(0) G^b_{\nu\sigma}(0) c_l^{\alpha'}(0)|\Omega\rangle \\
+ \langle\Omega|\bar{c}_i^\alpha(x) G^a_{\mu\rho}(x) s_j^\beta(x) \underline{G}^c_{\epsilon\tau}(y) B^\epsilon_d(y) B^\tau_e(y) \bar{s}_k^{\beta'}(0) G^b_{\nu\sigma}(0) c_l^{\alpha'}(0)|\Omega\rangle \Big).
\end{aligned} \tag{2.86}
$$

Due to the possible permutations arising from the three-gluon interaction, this calculation is significantly more involved. The above contractions lead to

$$
\begin{aligned}
\Pi^{\bar{q}\sigma Gq}_{\mu\nu}(q) = -\frac{i^3 g_s^3}{2} f^{fce} t^a_{\alpha\beta} t^b_{\beta'\alpha'} \int d^dx\, d^dy\, e^{iq\cdot x}\, \gamma^\rho_{ij} \gamma^\sigma_{kl} S^{m_c}_{li}(-x) \\
\times \Bigg(& \delta^{af}\delta^{be} H_{\mu\rho\epsilon\tau}(x-y)\Big(\frac{\partial}{\partial z^\nu} D_{\tau\sigma}(y-z) - \frac{\partial}{\partial z^\sigma} D_{\tau\nu}(y-z)\Big)|_{z=0} \\
& \times \langle \overline{s}^{\beta'}_k(0) B^c_\epsilon(y) s^\beta_j(x)\rangle \\
& + \delta^{af}\delta^{bc} H_{\mu\rho\epsilon\tau}(x-y)\Big(\frac{\partial}{\partial z^\nu} D_{\epsilon\sigma}(y-z) - \frac{\partial}{\partial z^\sigma} D_{\epsilon\nu}(y-z)\Big)|_{z=0} \\
& \times \langle \overline{s}^{\beta'}_k(0) B^e_\tau(y) s^\beta_j(x)\rangle \\
& + \delta^{ac}\delta^{be}\Big(\frac{\partial}{\partial x^\mu} D_{\rho\epsilon}(x-y) - \frac{\partial}{\partial x^\rho} D_{\mu\epsilon}(x-y)\Big) \\
& \times \Big(\frac{\partial}{\partial z^\nu} D_{\tau\sigma}(y-z) - \frac{\partial}{\partial z^\sigma} D_{\tau\nu}(y-z)\Big)|_{z=0} \\
& \times \langle \overline{s}^{\beta'}_k(0) G^f_{\epsilon\tau}(y) s^\beta_j(x)\rangle \\
& + \delta^{ac}\delta^{fb}\Big(\frac{\partial}{\partial x^\mu} D_{\rho\epsilon}(x-y) - \frac{\partial}{\partial x^\rho} D_{\mu\epsilon}(x-y)\Big) \\
& \times H_{\epsilon\tau\nu\sigma}(y-z)\langle \overline{s}^{\beta'}_k(0) B^e_\tau(y) s^\beta_j(x)\rangle \\
& + \delta^{ae}\delta^{bc}\Big(\frac{\partial}{\partial x^\mu} D_{\rho\tau}(x-y) - \frac{\partial}{\partial x^\rho} D_{\mu\tau}(x-y)\Big) \\
& \times \Big(\frac{\partial}{\partial z^\nu} D_{\epsilon\sigma}(y-z) - \frac{\partial}{\partial z^\sigma} D_{\epsilon\nu}(y-z)\Big)|_{z=0} \\
& \times \langle \overline{s}^{\beta'}_k(0) G^f_{\epsilon\tau}(y) s^\beta_j(x)\rangle \\
& + \delta^{ae}\delta^{fb}\Big(\frac{\partial}{\partial x^\mu} D_{\rho\tau}(x-y) - \frac{\partial}{\partial x^\rho} D_{\mu\tau}(x-y)\Big) \\
& \times H_{\epsilon\tau\nu\sigma}(y-z)\langle \overline{s}^{\beta'}_k(0) B^c_\epsilon(y) s^\beta_j(x)\rangle \Bigg).
\end{aligned}
\tag{2.87}
$$

We can apply the appropriate colour algebra identities (2.19) alongside (2.68) and (2.69) to simplify our non-local VEVs. Moving into momentum space, we substitute $y_\omega e^{ip\cdot y} = -i\frac{\partial}{\partial p_\omega} e^{ip\cdot y}$ (where p is a momentum parameter resulting from the introduction of Fourier transforms) and evaluate the traces and colour algebra to give us a much more simplified expression

$$\Pi_{\mu\nu}^{\overline{q}\sigma Gq}(q) = -\frac{3(d-2)g_s^2 m_c\langle\overline{s}\sigma\cdot Gs\rangle}{2d(d-1)}\int\frac{d^dk}{(2\pi)^d}\frac{(k\cdot k g_{\mu\nu}+(2d-3)k_\mu k_\nu)}{(k\cdot k)^2\left(m_c^2-(k-q)\cdot(k-q)\right)}. \tag{2.88}$$

Once we extract the vector component and apply TARCER to reduce the integrals, we get the expression

$$\begin{aligned}\Pi_{v}^{\overline{q}\sigma Gq}\left(q^2\right) =& \frac{3\ 2^{-d-3}(d-2)\pi^{-\frac{d}{2}}g_s^2 m_c\langle\overline{s}\sigma\cdot Gs\rangle}{(d-1)dq\cdot q}\\ &\times\left((3-2d)A^d_{\{1,m_c\}}+\left((2d-3)m_c^2+(2d+1)q\cdot q\right)B^d_{\{1,m_c\},\{1,0\}}\right).\end{aligned} \tag{2.89}$$

Once the master integrals in this expression are evaluated, we can then ϵ-expand this to obtain the final contribution to our OPE,

$$\begin{aligned}\Pi_{v}^{\overline{q}\sigma Gq}\left(q^2\right) =& -\frac{9ig_s^2 m_c\langle\overline{s}\sigma\cdot Gs\rangle)}{256\pi^2\epsilon}\\ &-\frac{ig_s^2 m_c\langle\overline{s}\sigma\cdot Gs\rangle}{512\pi^2 z^2}\left(18z^2\log\left(\frac{m_c^2}{\nu^2}\right)+2\left(9z^2-4z-5\right)\log(1-z)\right.\\ &\left.-z(31z+10)\right).\end{aligned} \tag{2.90}$$

Diagram 6 (IX)

This is the first of the mixed condensate diagrams arising from the $\overline{c}Bc$ interaction term in $\mathcal{L}_{int}$

$$\begin{aligned}\Pi_{\mu\nu}^{\overline{q}\sigma Gq}(q) =& -g_s^3\, t^a_{\alpha\beta}t^b_{\beta'\alpha'}t^c_{\gamma\delta}\int d^dx\, d^dy\, e^{iq\cdot x}\gamma^\rho_{ij}\gamma^\sigma_{kl}\gamma^\eta_{mn}\\ &\times\langle\Omega|\overline{c}_i^\alpha(x)G^a_{\mu\rho}(x)s_j^\beta(x)\overline{c}_m^\gamma(y)c_n^\delta(y)B^c_\eta(y)\overline{s}_k^{\beta'}(0)G^b_{\nu\sigma}(0)c_l^{\alpha'}(0)|\Omega\rangle\\ =& \frac{g_s^2\langle\overline{s}\sigma\cdot Gs\rangle}{48d(d-1)}\int d^dx\, d^dy\, y_\omega\, e^{iq\cdot x}\, H_{\mu\rho\nu\sigma}(x)\mathrm{tr}\left[\gamma^\rho\sigma^{\omega\eta}\gamma^\sigma S^{m_c}(-y)\gamma^\eta S^{m_c}(y-x)\right].\end{aligned} \tag{2.91}$$

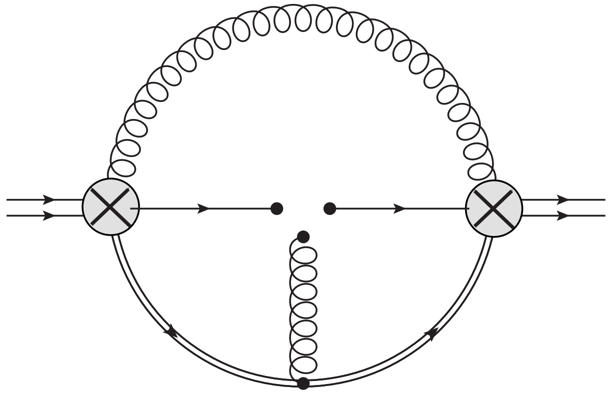

Figure 2.9: Mixed condensate contribution, diagram 6 (IX).

Moving to momentum space, substituting $y_\omega e^{ip\cdot y} = -i\frac{\partial}{\partial p_\omega} e^{ip\cdot y}$, and integrating by parts gives

$$\Pi_{\mu\nu}^{\bar{q}\sigma Gq}(q) = -\frac{ig_s^2\langle\bar{s}\sigma\cdot Gs\rangle}{48d(d-1)}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}\,\mathrm{tr}\left[\gamma^\rho(\sigma_{\omega\eta})\gamma^\sigma S^{m_c}(k-q)\gamma^\omega S^{m_c}(k-q)\gamma^\eta S^{m_c}(k-q)\right]$$
$$\times\left(g_{\rho\sigma}k_\mu k_\nu + g_{\mu\nu}k_\rho k_\sigma - g_{\mu\sigma}k_\rho k_\nu - g_{\rho\nu}k_\mu k_\sigma\right). \tag{2.92}$$

After extracting the vector component and applying TARCER, our expression becomes

$$\Pi_v^{\bar{q}\sigma Gq}\left(q^2\right) = -\frac{i2^{-d-4}(d-4)(d-2)\pi^{-\frac{d}{2}}g_s\langle\bar{s}\sigma\cdot Gs\rangle}{3dm_c q\cdot q}$$
$$\times\left(m_c^2\left(m_c^2-q\cdot q\right)\mathbf{B}^d_{\{1,m_c\},\{1,0\}} - \left(m_c^2+2q\cdot q\right)\mathbf{A}^d_{\{1,m_c\}}\right), \tag{2.93}$$

which evaluates and ϵ-expands to the finite expression

$$\Pi_v^{\bar{q}\sigma Gq}\left(q^2\right) = \frac{ig_s^2 m_c\langle\bar{s}\sigma\cdot Gs\rangle)}{256\pi^2}. \tag{2.94}$$

Diagram 7 (X)

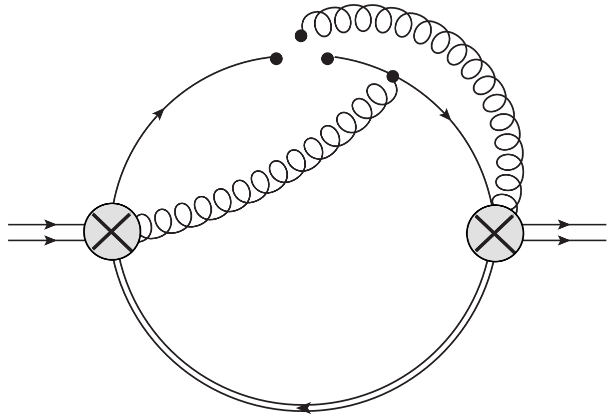

Figure 2.10: Mixed condensate contribution, diagram 7 (X).

Similar to the previous diagram, this is the first contribution to the mixed condensate arising from the $\overline{s}Bs$ interaction term in $\mathcal{L}_{int}$. Having one related mirrored diagram (an equivalent diagram resulting from a different permutation of field contractions), this contribution carries a multiplicity of 2, which has been included in (2.95)

$$\begin{aligned}
\Pi^{\bar{q}\sigma Gq}_{\mu\nu}(q) &= -2g_s^3\, t^a_{\alpha\beta} t^b_{\beta'\alpha'} t^c_{\gamma\delta} \int d^dx\, d^dy\, e^{iq\cdot x} \gamma^\rho_{ij}\gamma^\sigma_{kl}\gamma^\eta_{mn} \\
&\qquad \times \langle\Omega|\overline{c}^\alpha_i(x) G^a_{\mu\rho}(x) s^\beta_j(x) \overline{s}^\gamma_m(y) s^\delta_n(y) B^c_\eta(y) \overline{s}^{\beta'}_k(0) G^b_{\nu\sigma}(0) c^{\alpha'}_l(0)|\Omega\rangle \\
&= -\frac{ig_s^2\langle\overline{s}\sigma\cdot Gs\rangle}{12d(d-1)} \int d^dx\, d^dy\, e^{iq\cdot x} \left(\frac{\partial}{\partial x^\mu} D_{\rho\eta}(x-y) - \frac{\partial}{\partial x^\rho} D_{\mu\eta}(x-y)\right) \\
&\qquad \times \mathrm{tr}\left[S^{m_c}(-x)\gamma^\rho\sigma_{\nu\sigma}\gamma^\eta S^{m_s}(-y)\gamma^\sigma\right].
\end{aligned} \tag{2.95}$$

The TARCER-reduced vector component is then

$$\begin{aligned}
\Pi^{\bar{q}\sigma Gq}_v\left(q^2\right) &= \frac{2^{-d-2}(d-3)\pi^{-\frac{d}{2}} g_s^2 m_c \langle\overline{s}\sigma\cdot Gs\rangle}{3(d-1)dq\cdot q} \\
&\qquad \times \left(\left((d-2)m_c^2 + (d+2)q\cdot q\right) \mathbf{B}^d_{\{1,m_c\},\{1,0\}} - (d-2)\mathbf{A}^d_{\{1,m_c\}}\right),
\end{aligned} \tag{2.96}$$

and the final ϵ-expanded expression is

$$\Pi_v^{\overline{q}\sigma Gq}\left(q^2\right) = -\frac{ig_s^2 m_c\langle\overline{s}\sigma\cdot Gs\rangle}{384\pi^2\epsilon} - \frac{ig_s^2 m_c\langle\overline{s}\sigma\cdot Gs\rangle}{2304\pi^2 z^2}\left(6z^2\log\left(\frac{m_c^2}{\nu^2}\right) + \left(6z^2-4z-2\right)\log(1-z) - z(5z+2)\right). \tag{2.97}$$

Diagram 8 (XI)

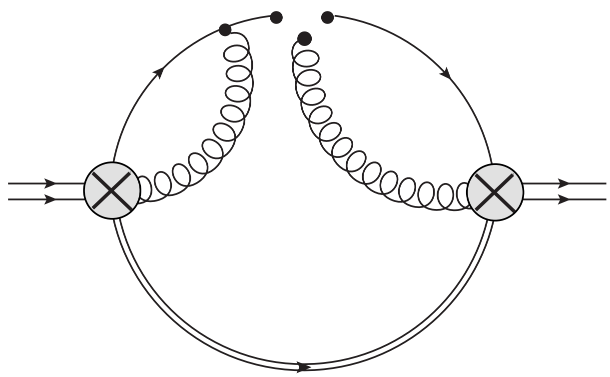

Figure 2.11: Mixed condensate contribution, diagram 8 (XI).

This contribution results again from the $\overline{s}Bs$ interaction term considered in the previous diagram; however, a careful examination of this diagram's topology reveals a massless tadpole (see equation (1.40)), which is a direct consequence of the massless approximation of the strange quark. As such, this diagram vanishes.

Diagram 9 (XII)

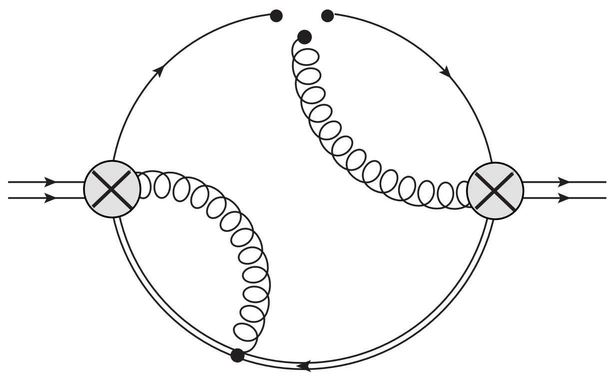

Figure 2.12: Mixed condensate contribution, diagram 9 (XII).

This diagram also arises from the $\overline{c}Bc$ interaction term in (2.9), and also introduces new complications into the calculation of the overall correlator. There is a multiplicity of two included in the calculation below to compensate for the similar mirrored diagram. We begin from the familiar set of contractions with the appropriate $\mathcal{L}_{int}$ term,

$$
\begin{aligned}
\Pi^{\overline{q}\sigma Gq}_{\mu\nu}(q) = -2g_s^3\, t^a_{\alpha\beta} t^b_{\beta'\alpha'} t^c_{\gamma\delta} \int d^d x\, d^d y\, e^{iq\cdot x} \gamma^\rho_{ij}\gamma^\sigma_{kl}\gamma^\eta_{mn} \\
\times \langle \Omega | \overline{c}^\alpha_i(x) G^a_{\mu\rho}(x) s^\beta_j(x) \overline{c}^\gamma_m(y) c^\delta_n(y) B^c_\eta(y) \overline{s}^{\beta'}_k(0) G^b_{\nu\sigma}(0) c^{\alpha'}_l(0) | \Omega \rangle \\
= \frac{2g_s^2 \langle \overline{s}\sigma\cdot Gs\rangle}{3d(d-1)} \int d^d x\, d^d y\, e^{iq\cdot x} \left(\frac{\partial}{\partial x^\mu} D_{\rho\eta}(x-y) - \frac{\partial}{\partial x^\rho} D_{\mu\eta}(x-y) \right) \\
\times \operatorname{tr}\left[\gamma^\rho \sigma_{\mu\rho}\gamma^\sigma S^{m_c}(-y)\gamma^\eta S^{m_c}(y-x)\right] \\
= \frac{2ig_s^2 \langle \overline{s}\sigma\cdot Gs\rangle}{3d(d-1)} \int \frac{d^d k}{(2\pi)^d} \frac{(k_\mu g_{\rho\eta} - k_\rho g_{\mu\eta})}{k^2} \operatorname{tr}\left[\gamma^\rho \sigma_{\mu\rho}\gamma^\sigma S^{m_c}(-q)\gamma^\eta S^{m_c}(k-q)\right].
\end{aligned}
\tag{2.98}
$$

Projecting out the vector component and applying recursion relations via TARCER, we obtain

$$
\begin{aligned}
\Pi_v^{\bar{q}\sigma Gq}\left(q^2\right) = -&\frac{2^{1-d}\pi^{-\frac{d}{2}}g_s^2 m_c\langle\bar{s}\sigma\cdot Gs\rangle}{3(d-1)dq\cdot q\left(M^2-q\cdot q\right)} \\
&\times\Bigg(\left((6-5d)q\cdot q-(d-2)m_c^2\right)\mathbf{A}^d_{\{1,m_c\}} \\
&\qquad + (d-2)\left(M^2-q\cdot q\right)^2\mathbf{B}^d_{\{1,m_c\},\{1,0\}}\Bigg).
\end{aligned}
\tag{2.99}
$$

The final ϵ-expanded expression we obtain is

$$
\begin{aligned}
\Pi_v^{\bar{q}\sigma Gq}\left(q^2\right) = -&\frac{ig_s^2 m_c(z-9)\langle\bar{s}\sigma\cdot Gs\rangle}{144\pi^2(z-1)\epsilon} \\
&-\frac{ig_s^2 m_c\langle\bar{s}\sigma\cdot Gs\rangle}{864\pi^2(z-1)z^2}\Bigg(6(z-9)z^2\log\left(\frac{m_c^2}{\nu^2}\right) \\
&\qquad + z\left(-13z^2+87z-6\right)+6(z-1)^3\log(1-z)\Bigg).
\end{aligned}
\tag{2.100}
$$

Examining the results of the ϵ-expansion in (2.100), we encounter a problem. Recall the Borel transform (1.42) discussed previously as the basis of the QCD sum rule analysis of the correlation function. Applying (1.42) to (2.100) would lead to an ϵ-dependence in our sum rule formulation (recall ϵ is a remnant of the dimensional regularization scheme used to regulate UV divergences), as the $\mathcal{O}\left(\epsilon^{-1}\right)$ term in (2.100) has a (simple) pole at $z=1$. The appearance of this divergence is problematic, as QCD is a renormalizable gauge theory which should not display dependence on this regularization parameter. Here, the field theory is communicating that there is a missing piece in our renormalization. There are two options available to look at: renormalize a parameter to absorb this divergence (such as a mass parameter or VEV), or examine composite operator renormalization of our hybrid current (2.7), and the resulting operator mixing.

If we eliminate this problematic z-dependence by renormalizing a parameter, we must examine additional $\mathcal{O}\left(g_s^2\right)$ terms, requiring the consideration of higher-order contributions. Our second option is to consider operator mixing. Given our quantum numbers $J^P=1^-$, there are other possible currents we can write with the same J^P, such as the conventional (i.e. non-hybrid) current

$$j_\mu^{\text{con}}(x) = m_c^2 \overline{c}^\alpha(x)\gamma_\mu s^\alpha(x). \tag{2.101}$$

Because this current is compatible with our desired quantum numbers, this conventional current could probe our states of interest and must be considered; hence what we are after is a renormalized composite current

$$j_\mu^{\text{renorm}}(x) = Z_1\, g_s \overline{c}_i^\alpha(x)\gamma_\rho t^a_{\alpha\beta} G^a_{\mu\rho}(x) s_j^\beta(x) + Z_2\, m_c^2 \overline{c}_i^\alpha(x)\gamma_\mu s_j^\beta(x) \tag{2.102}$$

where Z_1 and Z_2 are renormalization constants. If we exchange this renormalized current in our correlation function calculation, we see the modifications we need to make to our calculation of the correlation function

$$\begin{aligned}\langle\Omega|j_\mu^{\text{renorm}}(x) j_\nu^{\text{renorm}\dagger}(0)|\Omega\rangle = {} & |Z_1|^2 \langle\Omega|j_\mu(x) j_\nu{}^\dagger(0)|\Omega\rangle \\ & + Z_1 Z_2^\star \langle\Omega|j_\mu(x) j_\nu^{\text{con}\dagger}(0)|\Omega\rangle + Z_1^\star Z_2 \langle\Omega|j_\mu^{\text{con}}(x) j_\nu{}^\dagger(0)|\Omega\rangle \\ & + |Z_2|^2 \langle\Omega|j_\mu^{\text{con}}(x) j_\nu^{\text{con}\dagger}(0)|\Omega\rangle.\end{aligned} \tag{2.103}$$

The renormalization constant $Z_1 = 1 + \mathcal{O}(\alpha_s)$ associated with our hybrid current will have higher order α_s corrections; with the order of our calculation, we take the leading order contribution, $Z_1 = 1$. We can empirically determine what Z_2 should look like by examining the $\left(Z_1 Z_2^\star \langle\Omega|j_\mu(x) j_\nu^{\text{con}\dagger}(0)|\Omega\rangle + Z_1^\star Z_2 \langle\Omega|j_\mu^{\text{con}\dagger}(x) j_\nu(0)|\Omega\rangle\right)$ term in equation (2.103), and comparing against the discontinuity in (2.100) that is required to cancel. These contributions naturally form dimension five condensate diagrams of the topology depicted in Figure 2.13.

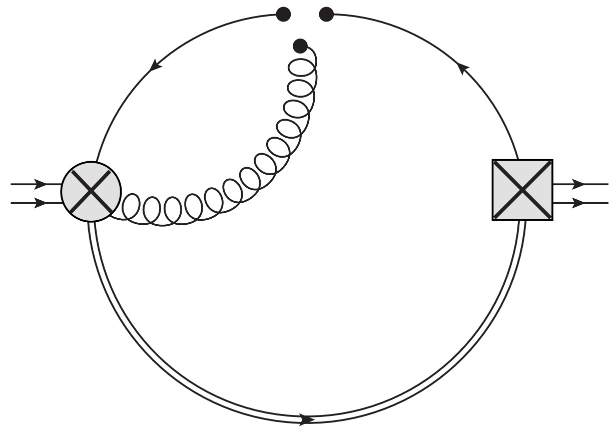

Figure 2.13: Feynman diagram representation of conventional and hybrid operator mixing. The square vertex represents the conventional current insertion (2.101).

The related mirrored diagram is not shown, but accounted for in the following calculation of the correlation functions

$$
\begin{aligned}
\Pi^{\text{mixing}}_{\mu\nu}(q) &= ig_s \int d^d x\, e^{iq\cdot x}\Big(Z_1 Z_2^\star \langle\Omega|\bar{c}_i^{\alpha}(x)\gamma^{\rho}_{ij} t^a_{\alpha\beta} G^a_{\mu\rho}(x) s^{\beta}_j(x)\bar{s}^{\alpha'}_k(0)\gamma_{\nu kl} c^{\alpha'}_l(0)|\Omega\rangle \\
&\qquad + Z_1^\star Z_2 \langle\Omega|\bar{c}_i^{\alpha}(x)\gamma_{\mu_{ij}} s^{\alpha}_j(x)\bar{s}^{\beta'}_k(0)\gamma^{\sigma}_{kl} t^b_{\beta'\alpha'} G^b_{\nu\sigma}(0) c^{\alpha'}_l(0)|\Omega\rangle\Big) \\
&= \frac{i\langle\bar{s}\sigma\cdot Gs\rangle}{4d(d-1)}\int d^d x e^{iq\cdot x} \\
&\qquad \times \Big(Z_1 Z_2^\star \mathrm{tr}[S^{m_c}(-x)\gamma^{\rho}\sigma_{\mu\rho}\gamma_{\nu}] + Z_1^\star Z_2 \mathrm{tr}[S^{m_c}(-x)\gamma_{\mu}\sigma_{\nu\sigma}\gamma^{\sigma}]\Big) \\
\Rightarrow \Pi^{\text{mixing}}_{v}(q^2) &= \frac{-2i\langle\bar{s}\sigma\cdot Gs\rangle}{d\, m_c(1-z)}\mathrm{Im}\,(Z_2)\,.
\end{aligned}
\tag{2.104}
$$

In the usual fashion, our projection (1.12) gives us the vector component in (2.104); as this diagram carries no closed loops it is unnecessary to apply TARCER to this diagram. Comparing this against the discontinuity that must be canceled, we can obtain information on the renormalization coefficients. The exact divergent contribution in (2.100) is determined to be

$$
-\frac{ig_s^2 m_c(z-9)\langle\bar{s}\sigma\cdot Gs\rangle}{144\pi^2(z-1)\epsilon} = -\frac{ig_s^2 m_c\langle\bar{s}\sigma\cdot Gs\rangle}{144\pi^2\epsilon} + \frac{ig_s^2 m_c\langle\bar{s}\sigma\cdot Gs\rangle}{18\pi^2(z-1)\epsilon}. \tag{2.105}
$$

The actual discontinuity is revealed above; this implies in our result (2.104) that (taking

$d \to 4 + 2\epsilon$),

$$\mathrm{Im}\,(Z_2) = \frac{(4+2\epsilon) g_s^2 m_c^2}{36\pi^2 \epsilon}. \tag{2.106}$$

Taking this factor, (2.104) becomes

$$\Pi_v^{\mathrm{mixing}}(q^2) = -\frac{i g_s^2 m_c \langle \bar{s}\sigma \cdot G s\rangle}{18\,\pi^2 (z-1)\epsilon}, \tag{2.107}$$

which (as expected), provides the appropriate cancellation needed to address the divergence in (2.100). Notice the factor of $\frac{1}{d}$ in (2.104) when combined with (2.106) cancels with the factor of $d = 4 + 2\epsilon$. This means that the contribution from Figure 2.13 only cancels the discontinuity shown in (2.105), and does not add any finite contributions. Combining (2.107) with our result (2.100) gives us a final expression with the discontinuity removed,

$$\begin{aligned}\Pi_v^{\bar{q}\sigma G q}\left(q^2\right) + \Pi_v^{\mathrm{mixing}}(q^2) = &-\frac{i g_s^2 m_c \langle \bar{s}\sigma \cdot G s\rangle}{144\pi^2 \epsilon} \\ &- \frac{i g_s^2 m_c \langle \bar{s}\sigma \cdot G s\rangle}{864\pi^2 (z-1) z^2} \Bigg(6(z-9) z^2 \log\left(\frac{m_c^2}{\nu^2}\right) + z\left(-13z^2 + 87z - 6\right) \\ &\qquad + 6(z-1)^3 \log(1-z) \Bigg).\end{aligned} \tag{2.108}$$

Note that at our particular order of calculation, the $|Z_2|^2$ term in (2.103) will not contribute; the renormalization-induced operator mixing occurs at $\mathcal{O}\left(\alpha_s\right)$ in Z_2, making any contribution of $|Z_2|^2$ a higher-order contribution. A thorough investigation of the composite operator renormalization very briefly discussed here may be examined in future work.

Diagram 10 (XIII)

It is important to note that the depiction of this condensate in Figure 2.14 is difficult to interpret in the context of conventional Feynman diagrams. The calculation of this diagram using our normal methodology is not possible as it introduces a zero momentum strange quark propagator, which diverges in our massless quark approximation. More explicitly,

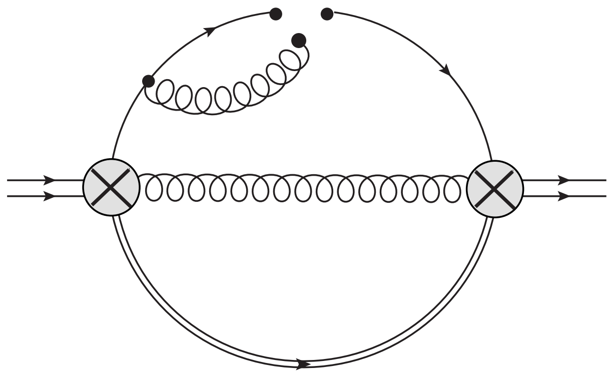

Figure 2.14: Mixed condensate contribution, diagram 10 (XIII).

Figure 2.14 is a result of the following contraction simplified into momentum space:

$$
\begin{aligned}
&- g_s^3\, t^a_{\alpha\beta} t^b_{\beta'\alpha'} t^c_{\gamma\delta} \int d^dx\, d^dy\, e^{iq\cdot x} \gamma^\rho_{ij} \gamma^\sigma_{kl} \gamma^\eta_{mn} \\
&\qquad\qquad \times \langle \Omega | \overline{c}^\alpha_i(x) G^a_{\mu\rho}(x) s^\beta_j(x) \overline{s}^\gamma_m(y) s^\delta_n(y) B^c_\eta(y) \overline{s}^{\beta'}_k(0) G^b_{\nu\sigma}(0) c^{\alpha'}_l(0) | \Omega \rangle \\
&= -\frac{4ig_s^2 \langle \overline{s}\sigma \cdot Gs \rangle}{24d(d-1)} \int \frac{d^dk d^dp d^dl}{(2\pi)^d} \delta^d(p-l-k+q) \delta^d(l) \\
&\qquad\qquad \times p_\omega \mathrm{tr}\left[\gamma^\eta \sigma^{\omega\eta} \gamma^\sigma S^{m_c}(p) \gamma^\omega S^{m_c}(p) \gamma^\rho S^{m_s}(l)\right]\, H_{\mu\rho\nu\sigma}(k).
\end{aligned}
\tag{2.109}
$$

We see that the zero-momentum propogator manifests in the delta function $\delta^d(l)$. It is conventional procedure to obtain contributions from diagrams such as these from higher-order terms in the series expansion of field operators appearing in other condensate calculations [16]. As such, we calculate this contribution from higher-order terms in the expansion of the quark field in the dimension four quark condensate, treating this diagram as a dimension five contribution to the previously calculated quark condensate (2.80). Revisiting the quark condensate result in (2.67), we recall that in the VEV we make a leading-order approximation and truncate the expansion at $\mathcal{O}(x^0)$. We can examine dimension five mixed condensate contributions to this by considering higher-order terms in the series expansion of $s^\beta_j(x)$. Elias *et al.* [26] gives, to lowest-order in quark mass m,

$$
\langle \overline{s}^{\beta'}_i(z) s^\beta_j(y) \rangle = \frac{\langle \overline{s}s \rangle}{12} \delta^{\beta\beta'} \delta_{ji} + \frac{\delta^{\beta'\beta} \langle \overline{s}\sigma \cdot Gs \rangle}{3} \left(-\frac{1}{96} y_\mu z_\nu \sigma^{\mu\nu}_{ji} - \frac{i}{64}(y-z)^2 \delta_{ij} \right). \tag{2.110}
$$

Notice that the leading order contribution is identical to (2.67). Inserting (2.110) into our original expression in equation (2.76),

$$
\begin{aligned}
\Pi^{\bar{q}\sigma\cdot Gq}_{\mu\nu}(q) &= ig_s^2 \int d^dx\, e^{iq\cdot x} \langle \Omega | \bar{c}_i^{\alpha}(x) t^a_{\alpha\beta} \gamma^\rho_{ij} G^a_{\mu\rho}(x) s^\beta_j(x) \bar{s}^{\beta'}_k(0) t^b_{\beta'\alpha'} \gamma^\sigma_{kl} G^b_{\nu\sigma}(0) c^{\alpha'}_l(0) | \Omega \rangle \\
&= -ig_s^2 (t^a t^a)_{\beta'\beta} \gamma^\sigma_{kl} \gamma^\rho_{ij} \int d^dx\, e^{iq\cdot x} H_{\mu\rho\nu\sigma}(x) S^{m_c}_{li}(-x) \langle \bar{s}^{\beta'}_k(0) s^\beta_j(x) \rangle \\
&= -\frac{4ig_s^2 \langle \bar{s}\sigma \cdot Gs \rangle}{3} \gamma^\sigma_{kl} \gamma^\rho_{ij} \int d^dx\, e^{iq\cdot x} H_{\mu\rho\nu\sigma}(x) S^{m_c}_{li}(-x) \left(-\frac{i}{64} x^2 \delta_{kj} \right).
\end{aligned}
\tag{2.111}
$$

In order to move this into momentum space, we first Fourier transform our propogators, and then substitute

$$
x^2 e^{ip\cdot x} = -g^{\omega\lambda} \frac{\partial^2}{\partial p^\omega \partial p^\lambda} e^{ip\cdot x}, \tag{2.112}
$$

and integrate by parts, such that we obtain

$$
\begin{aligned}
\Pi^{\bar{q}\sigma\cdot Gq}_{\mu\nu}(q) = -\frac{g_s^2 \langle \bar{s}\sigma \cdot Gs \rangle}{48} \int \frac{d^dk\, d^dp}{(2\pi)^d} \delta^d\left(p - (k-q)\right) g^{\omega\lambda} \mathrm{tr}\left[\left(\frac{\partial^2}{\partial p^\omega \partial p^\lambda} S^{m_c}(p) \right) \gamma^\rho \gamma^\sigma \right] \\
\times \times \frac{1}{k^2} (g_{\rho\sigma} k_\mu k_\nu + g_{\mu\nu} k_\rho k_\sigma - g_{\mu\sigma} k_\rho k_\nu - g_{\rho\nu} k_\mu k_\sigma).
\end{aligned}
\tag{2.113}
$$

With some careful evaluation and application of (2.15), we find that

$$
\frac{\partial^2}{\partial p^\omega \partial p^\lambda} S^{m_c}(p) = S^{m_c}(p) \gamma_\omega S^{m_c}(p) \gamma_\lambda S^{m_c}(p) + S^{m_c}(p) \gamma_\lambda S^{m_c}(p) \gamma_\omega S^{m_c}(p). \tag{2.114}
$$

The exchange of $\lambda \leftrightarrow \omega$ on the RHS of (2.114) simply results in a multiplicity of two (because of the $g^{\omega\lambda}$ in (2.113)). Thus,

$$
\begin{aligned}
\Pi^{\bar{q}\sigma\cdot Gq}_{\mu\nu}(q) = -\frac{g_s^3 \langle \bar{s}\sigma \cdot Gs \rangle}{24} \int \frac{d^dk}{(2\pi)^d} \mathrm{tr}\left[S^{m_c}(k-q) \gamma_\omega S^{m_c}(k-q) \gamma^\omega S^{m_c}(k-q) \gamma^\rho \gamma^\sigma \right] \\
\times \frac{1}{k^2} (g_{\rho\sigma} k_\mu k_\nu + g_{\mu\nu} k_\rho k_\sigma - g_{\mu\sigma} k_\rho k_\nu - g_{\rho\nu} k_\mu k_\sigma).
\end{aligned}
\tag{2.115}
$$

Projecting out the vector contribution and running through TARCER gives

$$
\Pi^{\bar{q}\sigma\cdot Gq}_{v}\left(q^2\right) = \frac{2^{-d-1}(d-2)\pi^{-\frac{d}{2}} g_s^2 m_c \langle \bar{s}\sigma \cdot Gs \rangle}{3dq\cdot q} \left(d\mathbf{A}^d_{\{1,m_c\}} - \left(dM^2 + (d-4) q\cdot q \right) \mathbf{B}^d_{\{1,m_c\},\{1,0\}} \right), \tag{2.116}
$$

and our final ϵ-expanded expression is

$$\Pi_v^{\bar{q}\sigma\cdot Gq}\left(q^2\right) = \frac{ig_s^2 m_c \langle \bar{s}\sigma\cdot Gs\rangle}{96\pi^2 z^2}((z-2)z + 2(z-1)\log(1-z)). \tag{2.117}$$

Diagram 11 (XIV)

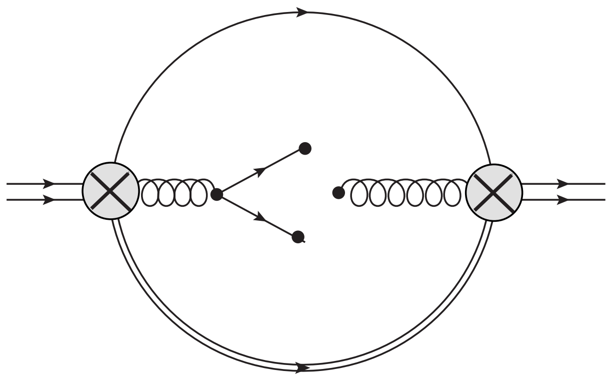

Figure 2.15: Mixed condensate contribution, diagram 11 (XIV).

As with the previous diagram containing zero-momentum propogator lines, we calculate the contributions of this diagram using an expansion of the previously addressed non-local VEV (2.44). In order to produce the correct quark content for a mixed condensate from the dimension four gluon condensate, we would likely look at the equation of motion (2.51). Recall however our discussion previously on the expansion of the dimension four gluon condensate. We argued in Section 2.4 that due to the Lorentz structure of the $\mathcal{O}(x)$ term in (2.44), we could not construct the local VEV using solely products of metrics; the $\mathcal{O}(x)$ term must be zero.

2.4.3 Dimension Six Condensate Contributions

In previous sum rule analyses of hybrid correlators, the contributions of the dimension six gluon condensates have been known to stabilize the end mass prediction [28, 29, 30, 31]. For our specific system, the dimension six condensates arise from not only (2.9), but from considering $\mathcal{O}(x^2)$ terms in the series expansion of the dimension four gluon condensate (2.66).

Diagram 12 (VI)

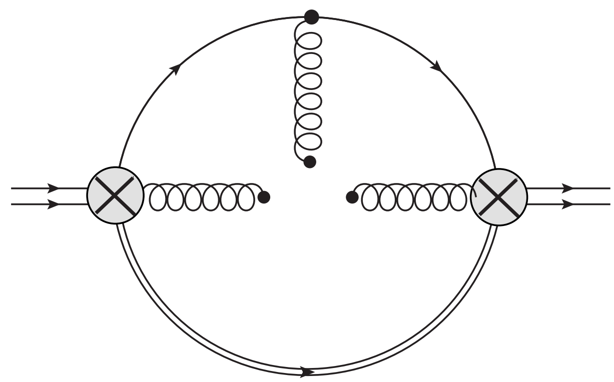

Figure 2.16: Dimension six condensate contribution, diagram 12 (VI) (strange flavour contribution).

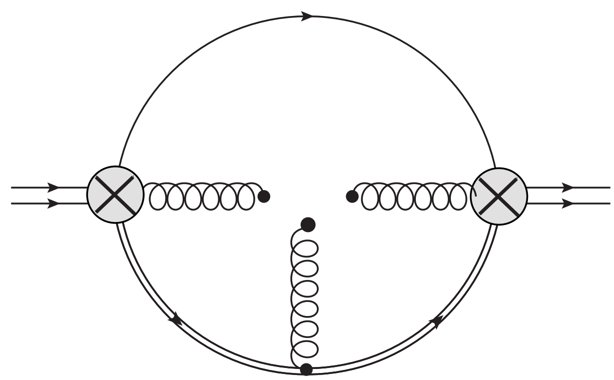

Figure 2.17: Dimension six condensate contribution, diagram 12 (VI) (charm flavour contribution).

Examining (2.9), we can pick out the terms that would naturally result in a set of three uncontracted gluon or gluon field strength tensor fields,

$$\mathcal{L}_{\text{int}} = \frac{1}{2} g_s \big(\big(\overline{s}_\alpha(x) \lambda^a_{\alpha\beta} \gamma^\mu s_\beta(x) B^a_\mu(x) + \overline{c}_\alpha(x) \lambda^a_{\alpha\beta} \gamma^\mu c_\beta(x) B^a_\mu(x) \big) + \ldots . \tag{2.118}$$

Because we must account for two flavours of quarks in our Lagrangian, this term produces two similar but distinct Feynman diagrams (Figures 2.16, 2.17). We will consider each separately for the sake of clarity. Beginning with the strange flavour term in (2.118), we can write out the appropriate contractions:

$$\Pi^{GGG}_{\mu\nu}(q) = ig_s^3 \int d^dx\, e^{iq\cdot x}\, t^a_{\alpha\beta} t^c_{\gamma\delta} t^b_{\beta'\alpha'}\, \gamma^\rho_{ij}\gamma^\lambda_{mn}\gamma^\sigma_{kl}$$
$$\times \langle\Omega|\overline{c}^\alpha_i(x) G^a_{\mu\rho}(x) s^\beta_j(x)\overline{s}^\gamma_m(y) s^\delta_n(y) B^c_\lambda(y)\overline{s}^{\beta'}_k(0) G^b_{\nu\sigma}(0) c^{\alpha'}_l(0)|\Omega\rangle, \tag{2.119}$$

leading to the simplified expression

$$\Pi^{GGG}_{\mu\nu}(q) = \frac{g_s^3}{2}\mathrm{tr}\left[t^c t^a t^b\right] \int d^dx\, d^dy\, e^{iq\cdot x}\, y^\lambda \mathrm{tr}\left[S^{m_s}(-y)\gamma^\beta S^{m_s}(y-x)\gamma^\rho S^{m_c}(x)\gamma^\omega\right]$$
$$\times \langle G^a_{\mu\rho}(x) G^c_{\lambda\beta}(y) G^b_{\nu\omega}(0)\rangle, \tag{2.120}$$

where we have used (2.6). Next, we move into momentum-space and rewrite the coordinate dependence y^λ as a derivative of an exponential term. Substituting the momentum-space propagators gives us

$$\Pi^{GGG}_{\mu\nu}(q) = \frac{g_s^3}{2}\mathrm{tr}\left[t^c t^a t^b\right] \int \frac{d^dx\, d^dy\, d^dp_1\, d^dp_2\, d^dp_3}{(2\pi)^{3d}}\, e^{iq\cdot x}\, e^{i(p_3-p_2)\cdot x}\, e^{i(p_2-p_1)\cdot y}$$
$$\times y^\lambda \mathrm{tr}\left[S^{m_s}(p_1)\gamma^\beta S^{m_s}(p_2)\gamma^\rho S^{m_c}(p_3)\gamma^\omega\right] \langle G^a_{\mu\rho}(x) G^c_{\lambda\beta}(y) G^b_{\nu\omega}(0)\rangle. \tag{2.121}$$

We can then substitute the coordinate-dependence y^λ for a derivative of the resulting exponent terms

$$y^\lambda e^{i(p_2-p_1)\cdot y} = i\frac{\partial}{\partial p_{1\lambda}}\, e^{i(p_2-p_1)\cdot y}, \tag{2.122}$$

and finally integrate by parts to move the partial derivative to $S^{m_s}(p_1)$ (which we can evaluate using (2.15)), giving us an intermediate result of

$$\Pi^{GGG}_{\mu\nu}(q) = -\frac{g_s^3}{2}\mathrm{tr}\left[t^c t^a t^b\right] \int \frac{d^dp_1}{(2\pi)^d} \mathrm{tr}\left[S^{m_s}(p_1)\gamma^\lambda S^{m_s}(p_1)\gamma^\beta S^{m_s}(p_1)\gamma^\rho S^{m_c}(p_1-q)\gamma^\omega\right]$$
$$\times \langle G^a_{\mu\rho}(x) G^c_{\lambda\beta}(y) G^b_{\nu\omega}(0)\rangle. \tag{2.123}$$

Substituting identities from (2.19) and (2.70), the vector component of this contribution to the correlator is

$$
\begin{aligned}
\Pi_v^{GGG}(q) =& \frac{i 2^{-d} \pi^{-\frac{d}{2}} \langle g^3 G^3 \rangle}{2d(d-1)q^2\left(m_c^2 - q^2\right)} \\
& \times \Bigg(\left(((d-5)d+7)q^2 - m_c^2\right) \mathbf{A}^d_{\{1,m_c\}} \\
& \qquad - \left((d-4)(d-3)m_c^2 q^2 + ((d-5)d+7)q^4 + m_c^4\right) \mathbf{B}^d_{\{1,m_c\}\{1,0\}} \Bigg).
\end{aligned}
\tag{2.124}
$$

We carefully note that while our colour algebra $\mathrm{tr}\left[t^c t^a t^b\right] = \frac{1}{4}\left(d^{cab} + i f^{cab}\right)$ carries a contribution from the symmetric structure constants d^{cab}, as the condensate must exhibit Bose symmetry (being constructed of gluon fields), and the VEV $\langle G^a_{\beta\lambda} G^b_{\lambda\mu} G^c_{\mu\beta} \rangle$ is asymmetric under index exchange, the only contribution must come from the antisymmetric structure constants f^{cab}. Subsequent evaluation of the integrals, application of renormalization scheme, and ϵ-expansion gives a final result of

$$
\Pi_v^{GGG}\left(q^2\right) = -\frac{\langle g^3 G^3 \rangle}{128\pi^2 \epsilon} - \frac{\langle g^3 G^3 \rangle \left(6z^2 \log\left(\frac{m_c^2}{\nu^2}\right) - 7z^2 + 2\left(3z^2 - 1\right)\log(1-z) - 2z\right)}{768\pi^2 z^2}.
\tag{2.125}
$$

The treatment of the charm-flavoured interaction term (depicted in Figure 2.17) proceeds similarly,

$$
\begin{aligned}
\Pi_{\mu\nu}^{GGG}(q) &= i g_s^3 \int d^d x \, e^{iq\cdot x} \, t^a_{\alpha\beta} t^c_{\gamma\delta} t^b_{\beta'\alpha'} \, \gamma^\rho_{ij} \gamma^\lambda_{mn} \gamma^\sigma_{kl} \\
& \qquad \times \langle \Omega | \overline{c}^\alpha_i(x) G^a_{\mu\rho}(x) s^\beta_j(x) \overline{c}^\gamma_m(y) c^\delta_n(y) B^c_\lambda(y) \overline{s}^{\beta'}_k(0) G^b_{\nu\sigma}(0) c^{\alpha'}_l(0) | \Omega \rangle \\
&= -\frac{g_s^3}{2} \mathrm{tr}\left[t^a t^c t^b\right] \int \frac{d^d p}{(2\pi)^d} \mathrm{tr}\left[S^{m_c}(p-q)\gamma^\lambda S^{m_c}(p-q)\gamma^\beta S^{m_c}(p-q)\gamma^\rho S^{m_s}(p)\gamma^\omega\right] \\
& \qquad \times \langle G^a_{\mu\rho}(x) G^c_{\lambda\beta}(y) G^b_{\nu\omega}(0) \rangle,
\end{aligned}
\tag{2.126}
$$

giving us the vector component

$$\Pi_v^{GGG}\left(q^2\right) = -\frac{i2^{-d}\pi^{-\frac{d}{2}}\langle g^3G^3\rangle}{2d(d-1)q^2}\left(\mathbf{A}^d_{\{1,m_c\}} + \left(((d-5)d+7)q^2 - m_c^2\right)\mathbf{B}^d_{\{1,m_c\}\{1,0\}}\right). \quad (2.127)$$

This is expanded and expressed in its final form as

$$\Pi_v^{GGG}\left(q^2\right) = -\frac{\langle g^3G^3\rangle}{128\pi^2\epsilon} - \frac{\langle g^3G^3\rangle\left(6z^2\log\left(\frac{m_c^2}{\nu^2}\right) + 2\left(3z^2-4z+1\right)\log(1-z) - (7z-2)z\right)}{768\pi^2z^2}. \quad (2.128)$$

Diagram 13 (V)

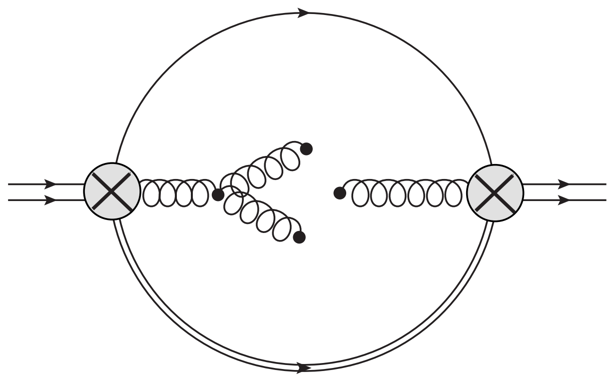

Figure 2.18: Dimension six condensate contribution, diagram 13 (V).

The final contribution due to the dimension six gluon condensate arises from considerations of the $\mathcal{O}\left(x^2\right)$ term in the series expansion of our non-local VEV (2.44). Recall that we had previously determined the form of this VEV in (2.70). We proceed with the calculation identically to our treatment of the dimension four gluon condensate starting from (2.73), only redefined in terms of $G_{\mu\nu}$ instead of $G^a_{\mu\nu}$

$$\begin{aligned}
\Pi^{\mathrm{GG,GGG}}_{\mu\nu}\left(q\right) &= i\int \frac{d^dp}{\left(2\pi\right)^d}\mathrm{tr}\left[S^{m_c}\left(p+q\right)\gamma^\sigma S^{m_s}\left(p\right)\gamma^\rho\right]\langle G_{\mu\rho}(x)G_{\nu\sigma}(0)\rangle \\
&= i\int d^dx\, e^{iq\cdot x}\mathrm{tr}\left[S^{m_c}\left(x\right)\gamma^\sigma S^{m_s}\left(-x\right)\gamma^\rho\right]\langle G_{\mu\rho}(x)G_{\nu\sigma}(0)\rangle \\
\Rightarrow \Pi^{GGG}_{\mu\nu} &= \frac{i}{2}\int d^dx\, x^{\omega_1}x^{\omega_2}\, e^{iq\cdot x}\mathrm{tr}\left[S^{m_c}\left(x\right)\gamma^\sigma S^{m_s}\left(-x\right)\gamma^\rho\right] \\
&\qquad \times \langle\left[D_{\omega_1}(0),\left[D_{\omega_2}(0),G_{\mu\rho}(0)\right]\right]G_{\nu\sigma}(0)\rangle.
\end{aligned} \tag{2.129}$$

After moving into momentum-space, we can once again make a substitution for the coordinate-dependence x^ω_1 and x^ω_2,

$$\begin{aligned}
x^\omega_1 e^{ip\cdot x} &= -i\frac{\partial}{\partial p_{\omega_1}}e^{ip\cdot x} \\
x^\omega_2 e^{-ik\cdot x} &= i\frac{\partial}{\partial k_{\omega_2}}e^{-ik\cdot x}
\end{aligned} \tag{2.130}$$

which, after integrating by parts and evaluating the derivatives, leads to a final expression

$$\begin{aligned}
\Pi_{\mu\nu}^{GGG} =& \frac{i}{4}\int \frac{d^d k}{(2\pi)^d} \mathrm{tr}\left[S^{m_c}(k-q)\gamma^{\omega_1} S^{m_c}(k-q)\gamma^{\rho} S^{m_s}(k)\gamma^{\omega_2} S^{m_s}(k)\gamma^{\sigma}\right] \\
&\times \Bigg(- \frac{i\langle g^3 G^3\rangle}{d(d-2)(d+2)}\, g_{\omega_1\omega_2}\left(g_{\mu\nu}g_{\rho\sigma} - g_{\rho\nu}g_{\mu\sigma}\right) \\
&- \frac{i\langle g^3 G^3\rangle}{2d(d-2)(d+2)}\left[g_{\omega_2\nu}\left(g_{\omega_1\mu}g_{\rho\sigma} - g_{\omega_1\rho}g_{\mu\sigma}\right) - g_{\omega_2\sigma}\left(g_{\omega_1\mu}g_{\rho\nu} - g_{\omega_1\rho}g_{\mu\nu}\right)\right] \\
&+ \frac{3i\langle g^3 G^3\rangle}{2d(d-1)(d-2)(d+2)}\left[g_{\omega_1\nu}\left(g_{\omega_2\mu}g_{\rho\sigma} - g_{\omega_2\rho}g_{\mu\sigma}\right) - g_{\omega_1\sigma}\left(g_{\omega_2\mu}g_{\rho\nu} - g_{\omega_2\rho}g_{\mu\nu}\right)\right]\Bigg),
\end{aligned} \tag{2.131}$$

where we have expressed our local VEV in terms of the dimension six gluon condensate using our previously derived (2.71). The resulting expansion is messy, but upon application of TARCER gives us a result in terms of our reference integrals

$$\begin{aligned}
\Pi_{v}^{GGG} =& \frac{2^{-d-2}(d-2)\pi^{-\frac{d}{2}}\langle g^3 G^3\rangle}{(d-1)d(d+2)q^2\left(m_c^2-q^2\right)} \\
&\times \Big(\left(-2\left(2d^2-13d+20\right)m_c^2q^2 + \left(-4d^2+17d-22\right)q^4 + (d+2)m_c^4\right)\mathbf{B}^d_{\{1,m_c\},\{1,0\}} \\
&\quad - \left(\left(-4d^2+17d-22\right)q^2 + (d+2)m_c^2\right)\mathbf{A}^d_{\{1,m_c\}}\Big),
\end{aligned} \tag{2.132}$$

which leads us to the ϵ-expanded expression

$$\Pi_{v}^{GGG} = \frac{\langle g^3 G^3\rangle}{64\pi^2\epsilon} + \frac{\langle g^3 G^3\rangle\left(6z^2\log\left(\frac{m_c^2}{\nu^2}\right) + 2(3z^2-1)\log(1-z) - (5z+2)z\right)}{384\pi^2 z^2}. \tag{2.133}$$

Chapter 3

Summary of Results and Future Outlook

3.1 Summary of Results

With the calculation of the correlation function complete, we can now combine our perturbative and non-perturbative results to form the final correlator. The following is a summary of the previously calculated contributions organized by condensate dimension. Here we have expressed our results in terms of the strong coupling constant $\alpha_s = \frac{g_s^2}{4\pi}$, the renormalization scale ν, and the dimensionless quantity $z = \frac{q^2}{m_c^2}$.

3.1.1 Perturbation Theory Results

For perturbation theory to $\mathcal{O}(m_s)$ in our light quark mass expansion, our result takes the form

$$\Pi_v^{\text{pert}} = \Pi_0 + m_s \Pi_1. \tag{3.1}$$

By including the $\mathcal{O}(m_s)$ correction we may examine the particular case of an open-strange correlator, and taking the chiral limit $m_s = 0$ we can look at the case of the heavy-light correlator.

The coefficients in (3.1) are

$$
\begin{aligned}
\Pi_0 = {} & \frac{\alpha m_c^6(2z+1)}{32\pi^3\epsilon^2} - \frac{\alpha m_c^6\left(-120\log\left(\frac{m_c^2}{\nu^2}\right) - 240z\log\left(\frac{m_c^2}{\nu^2}\right) + 4z^3 - 45z^2 + 180z + 150\right)}{1920\pi^3\epsilon} \\
& + \frac{\alpha m_c^6}{345600\pi^3 z^2}\Bigg(- 360z^2\left(4z^3 - 45z^2 + 180z + 150\right)\log\left(\frac{m_c^2}{\nu^2}\right) \\
& \qquad + 7200(6z+3)z^2\left(\log^2\left(\frac{m_c^2}{\nu^2}\right) + \mathrm{Li}_2(z)\right) + 4212z^5 - 44865z^4 \\
& \qquad + 240z^3\left(45\pi^2 - 53 + 180\log(2)\right) + 360\left(157 + 15\pi^2\right)z^2 \\
& \qquad - 360z - 360(1-z)(z^3(41-4z) + 141z^2 + z + 1)\log(1-z)\Bigg),
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
\Pi_1 = \frac{\alpha m_c^5}{48\pi^3}\Bigg(& - \frac{z}{\epsilon^2} - \frac{z\left(24\log\left(\frac{m_c^2}{\nu^2}\right) + 3z - 22\right)}{12\epsilon} \\
& - 2z\mathrm{Li}_2(z) + \frac{(1-z)(1-z(5-z(3z+13)))\log(1-z)}{6z^2} \\
& - \frac{\left(z^2\left(-225z + 72\pi^2 + 116\right) + 132z - 24\right)}{144z} \\
& - 2z\log^2\left(\frac{m_c^2}{\nu^2}\right) + \frac{z(22-3z)}{6}\log\left(\frac{m_c^2}{\nu^2}\right)\Bigg).
\end{aligned} \tag{3.3}
$$

3.1.2 Dimension Four Condensate Results

For the dimension four condensate parameterized by $m_c\langle\bar{s}s\rangle$ and $\langle\alpha G^2\rangle$,

$$
\begin{aligned}
\Pi_v^{\mathrm{4D}} = {} & \frac{\alpha m_c^3\,(z-9)\,\langle\bar{s}s\rangle}{18\pi\epsilon} + \frac{m_c^2\,(9-4z)\,\langle\alpha G^2\rangle}{144\pi\epsilon} \\
& + \frac{\alpha m_c^3\langle\bar{s}s\rangle\left(3(z-9)z^2\log\left(\frac{m_c^2}{\nu^2}\right) - 5z^3 + 30z^2 - 3z + 3(z-1)^3\log(1-z)\right)}{54\pi z^2} \\
& + \frac{m_c^2\langle\alpha G^2\rangle\left(-6(4z-9)z^2\log\left(\frac{m_c^2}{\nu^2}\right) + 44z^3 - 87z^2 + 6z - 6(z-1)^2(4z-1)\log(1-z)\right)}{864\pi z^2}.
\end{aligned} \tag{3.4}
$$

3.1.3 Dimension Five Condensate Results

Summarized here are the combined contributions to the dimension five "mixed" condensate parameterized by $\langle \bar{s}\sigma \cdot Gs \rangle$. Included is the contribution (2.107) from the renormalization-induced operator mixing diagram necessary to cancel off our unregulated divergence in Figure 2.12:

$$
\begin{aligned}
\Pi_v^{5\mathrm{D}} = &-\frac{19i\alpha m_c \langle \bar{s}\sigma \cdot Gs \rangle}{144\pi\epsilon} + \frac{i\alpha m_c \left(569z^2 - 1191z + 78\right) \langle \bar{s}\sigma \cdot Gs \rangle}{1728\pi(z-1)z} \\
&- \frac{i\alpha m_c (19z-51) \log\left(\frac{M^2}{\nu^2}\right) \langle \bar{s}\sigma \cdot Gs \rangle}{144\pi(z-1)} - \frac{i\alpha m_c (z-1)(38z-13)\log(1-z) \langle \bar{s}\sigma \cdot Gs \rangle}{288\pi z^2}.
\end{aligned}
\tag{3.5}
$$

3.1.4 Dimension Six Gluon Condensate Results

Finally, the combined contributions to the dimension six gluon condensates parameterized by $\langle g^3 G^3 \rangle$ are as follows:

$$
\Pi_v^{6\mathrm{D}} = \frac{((z-1)z + (2z-1)\log(1-z))\langle g^3 G^3 \rangle}{192\pi^2 z^2}. \tag{3.6}
$$

3.2 Future Directions

3.2.1 Sum Rules and Mixing

Calculation of the correlation function enables us to extract mass predictions using methods such as QCD sum rules (discussed briefly in Chapter 1). We are particularly interested in open-flavour systems with a $J^P = 1^-$, and as the quantum numbers defining our system are not explicitly exotic (i.e., they are not unique to hybrid systems), there is a possibility of coupling of conventional open-flavour states of the same quantum number to our hybrid current. This complicates the sum rule analysis of the system, which could be addressed in a number of ways. We previously discussed the general formulation of QCD sum rules in Chapter 1, where we modeled the hadronic spectral function with the narrow resonance model given by (1.50). In this expression, f_r represents the coupling of a particular state characterized by the mass m_r to our hybrid current; for states carrying exotic quantum numbers, we can eliminate the possibility of the current coupling to conventional states. In our case, where we are using a hybrid current that carries conventional quantum numbers, the possibility emerges that conventional states may couple to the current in addition to the hybrid states, in which case our sum rule picks up a more complex structure

$$R_0(\tau, s_0) = \pi f_{\bar{c}Gs}^2 m_{\bar{c}Gs}^2 + \pi f_{\bar{c}s}^2 m_{\bar{c}s}^2. \tag{3.7}$$

The addition of the conventional coupling term complicates the sum rules analysis; though the mass of the conventional charm-strange (D_s) and charm-light (D_0) mesons are well-known, the values for the couplings $f_{\bar{c}s}$ and $f_{\bar{c}Gs}$ are not.

Higher-weighted Sum Rules

In the case of a single coupling, QCD sum rules utilizes different weightings of the sum rule expression to solve for the hadronic mass prediction

$$\frac{-\frac{d}{d\tau} R_0(\tau, s_0)}{R_0(\tau, s_0)} = \frac{\mathcal{L}^{-1}\left[s\,\Pi\left(Q^2\right)\right] - \frac{1}{\pi}\int_{s_0}^{\infty} ds\, s\, \mathrm{Im}\Pi^{\mathrm{QCD}}\left(s\right) e^{-s\tau}}{\mathcal{L}^{-1}\left[\Pi\left(Q^2\right)\right] - \frac{1}{\pi}\int_{s_0}^{\infty} ds\, \mathrm{Im}\Pi^{\mathrm{QCD}}\left(s\right) e^{-s\tau}} = m_H^2. \tag{3.8}$$

In the case of multiple couplings, the expression (3.8) is no longer sufficient to isolate m_H. One possibility is to add additional weights of sum rules in order to eliminate the extra coupling constant (as the additional mass terms corresponding to conventional states are known). However, in adding higher weights of momentum ($s = -Q^2$) to our sum rule, we wash out information in our correlator; more terms in our correlation function fall victim to the Borel transform in Q^2. Whether or not the next-highest weight of sum rule leaves enough information in our leading order correlator requires further investigation.

Analyzing the Mass as a Function of the Coupling

Our previous investigations of hybrid systems with conventional quantum numbers have utilized a different technique to examine the dependence of the hadronic mass to the conventional state coupling [28]. Instead of attempting to eliminate the coupling term, the hadronic mass is instead examined as a function of this conventional coupling. In the case of our open-charm system, we may write

$$m^2_{\bar{c}Gs} = \frac{R_1(\tau, s_0) - f^2_{\bar{c}s} m^4_{\bar{c}s} e^{-m^2_{\bar{c}s}\tau}}{R_0(\tau, s_0) - f^2_{\bar{c}s} m^2_{\bar{c}s} e^{-m^2_{\bar{c}s}\tau}}. \tag{3.9}$$

From here, we may evaluate the hybrid mass $m_{\bar{c}Gs}$ and examine its variation as the coupling $f_{m_{\bar{c}s}}$ increases. Though this method does not give a definite mass prediction, it allows a definite lower bound to the hybrid mass, as well as insight into the mixing dynamics that may allow for conservative estimation of the hybrid mass.

Implementing Gaussian Sum Rules

Finally, we might examine sum rule techniques outside of the Laplace sum rules presented. Gaussian sum rules (GSR) have been shown to be particularly useful for examining mixed states [32]; their formulation is similar to that of the Laplace sum rules, only with a Gaussian kernel replacing the exponential kernel present in Laplace sum rules.

3.2.2 Other Systems

It should be mentioned that although our study of the charm-strange and charm-light hybrid system focused on a particular flavour structure, the correlator should be valid for any heavy-light system; analysis of other heavy-light systems such as bottom-strange or bottom-light systems are equally accessible using the results of our correlator. As well, modification of the hybrid current (2.7) or application of the scalar projection in (1.12) allows us to explore other quantum numbers in the mass spectrum. For example, by modifying our current to

$$j_\mu = g_s \bar{c} t^a \gamma^\nu \gamma_5 G^a_{\mu\nu} s \tag{3.10}$$

we change the parity from our original current and probe now the $J^P = 1^+$ states of the heavy-light hybrid system.

3.3 Concluding Remarks

The correlation function for the heavy-light hybrid system corresponding to the current $j_\mu = g_s \bar{c} t^a \gamma^\nu G^a_{\mu\nu} s$ and $J^P = 1^-$ was calculated, and consequences of renormalization-induced operator mixing were highlighted. The results presented here open the doors for future open-flavour hybrid mass predictions, provided the complications from possible couplings to conventional open-flavour states are addressed. We suggest several methods of addressing this in future studies and envision that once addressed, a comprehensive analysis of the heavy-light system for all possible quantum numbers may be possible in a fashion similar to that of previous studies [28]. With novel hadronic structures beginning to expand the landscape of QCD (such as recent tetraquark [33, 34] and pentaquark [35] discoveries), and with the availability of experimental facilities such as FAIR and GlueX with the capacity to thoroughly explore the entire quarkonia (closed-flavour) spectrum [9, 10], it is important now more than ever to accurately predict the properties of novel systems such as the hybrid meson so that we have the means to interpret the experimental data obtained at these facilities, and others around the world. As our ability to probe the internal structure of mesons increases, so does the demand for theoretical calculations to support experimental findings.The existence of hybrids would reinforce our understanding of the strong interaction and of colour confinement; likewise, the absence of hybrids would lead us towards a more complete description of the strong force. The exploration of exotic structures in QCD will, without question, lead to a better understanding of the interactions which govern the structure and formation of matter.

References

[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

[2] G. Zweig and S. An, CERN-TH-412 (1964).

[3] O. W. Greenberg, Phys. Rev. Lett. **13**, 598 (1964).

[4] M. Y. Han and Y. Nambu, Phys. Rev. **139**, B1006 (1965).

[5] E. D. Bloom *et al.*, Phys. Rev. Lett. **23**, 930 (1969).

[6] D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973).

[7] J. Beringer *et al.*, Phys. Rev. D **86** (2012), 0104147.

[8] C. Meyer and E. Swanson, Prog. Part. Nucl. Phys. **82**, 21 (2015).

[9] D. S. Carman, AIP Conf. Proc. **814**, 173 (2006).

[10] M. Destefanis, Nucl. Phys. B - Proc. Sup. **245**, 199 (2013), 0710.5664.

[11] M. E. Peskin, *An Introduction to Quantum Field Theory* (Addison-Wesley Publishing Company, 1995).

[12] J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).

[13] K. Olive and Particle Data Group, Chinese Phys. **090001**, 1 (2014).

[14] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968).

[15] S. Narison, *QCD as a Theory of Hadrons: From Partons to Confinement* (Cambridge University Press, 2004).

[16] P. Pascual and R. Tarrach, *QCD: Renormalization for the Practitioner*, Lecture Notes in Physics Vol. 194 (Springer Berlin Heidelberg, Berlin, Heidelberg, 1984).

[17] R. Mertig and R. Scharf, Comput. Phys. Commun. **111**, 13 (1998), 9801383.

[18] D. J. Broadhurst, J. Fleischer, and O. V. Tarasov, Z. Phys. C Part. Fields **60**, 287 (1993), 9304303.

[19] E. E. Boss and A. I. Davydychev, Theor. Math. Phys. **89**, 1052 (1991).

[20] A. I. Davydychev, J. Math. Phys. **33**, 358 (1992).

[21] M. Shifman, Prog. Theor. Phys. Sup. **131**, 80 (1998), 9802214.

[22] M. Shifman, A. Vainshtein, and V. Zakharov, Nucl. Phys. B **147**, 519 (1979).

[23] E. Bagan, M. R. Ahmady, V. Elias, and T. G. Steele, Z. Phys. C Part. Fields **61**, 157 (1994).

[24] D. Binosi and L. Theußl, Comput. Phys. Commun. **161**, 76 (2004), 0309015.

[25] J. Latorre, P. Pascual, and S. Narison, Z. Phys. C Part. Fields **34**, 347 (1987).

[26] V. Elias, T. G. Steele, and M. D. Scadron, Phys. Rev. D **38**, 1584 (1988).

[27] J. Govaerts, L. Reinders, H. Rubinstein, and J. Weyers, Nucl. Phys. B **258**, 215 (1985).

[28] W. Chen *et al.*, J. High Energy Phys. **2013** (2013).

[29] C.-F. Qiao, L. Tang, G. Hao, and X.-Q. Li, J. Phys. G Nucl. Partic. **39**, 15005 (2012).

[30] D. Harnett, R. T. Kleiv, T. G. Steele, and H.-y. Jin, J. Phys. G Nucl. Partic. **39**, 125003 (2012), 1206.6776.

[31] R. Berg, D. Harnett, R. T. Kleiv, and T. G. Steele, Phys. Rev. D **86**, 034002 (2012).

[32] D. Harnett, R. T. Kleiv, K. Moats, and T. G. Steele, Nucl. Phys. A **850**, 110 (2011).

[33] BESIII Collaboration, M. Ablikim *et al.*, Phys. Rev. Lett. **110**, 252001 (2013).

[34] Belle Collaboration, Z. Q. Liu *et al.*, Phys. Rev. Lett. **110**, 252002 (2013).

[35] LHCb Collaboration, R. Aaij *et al.*, Phys. Rev. Lett. **115**, 072001 (2015).

Appendix A

Appendix A: Mathematica Code

A.1 Mathematica Code

The following is the Mathematica code I have written to evaluate traces of Dirac structures appearing in calculating the correlator of hybrid currents. The code includes procedures for implementing TARCER [24], Feynman diagram vertices and propogators, as well as non-perturbative VEV expressions for evaluation of condensate diagrams.

Master Sheet

Reset

In[1]:=
```
Clear[diracMatrix, momentum, momentumConvert, momentumUnConvert,
 tr, g, lam, Levi, GluProp, FermProp, GhostProp, GluVert3, GluVert4,
 FermVert, GhostVert, GBVec, GBSca, DDGVacuumFactor, GVacuumFactor,
 DDFermProp, DFermProp, Current1, Current2, G5Current1, G5Current2,
 GTwidVacuumFactor, DDGTwidVacuumFactor, f, dSym, r, free1, free2, free3,
 free4, free5, free6, μ, ν, a, b, G, Tarce, expandPFQ, evalInt, sigma, δ]
```

Identities

Assumptions

In[2]:=
```
$Assumptions = d ∈ Complexes;
$Assumptions = m ∈ Reals;
$Assumptions = M ∈ Reals;
$Assumptions = gs ∈ Reals;
Protect[{γ, m, M, gs, d}];
```

Kronecker Delta

In[7]:=
```
δ[μ_, ν_] /; μ ∈ Integers && ν ∈ Integers := Piecewise[{μ = ν, 1}, 0]
δ /: j_[a___, μ_, b___, ν_, c___] δ[μ_, ρ_] /; FreeQ[j, diracMatrix] := j[a, ρ, b, ν, c]
δ /: j_[a___, μ_, b___, ν_, c___] δ[ρ_, μ_] /; FreeQ[j, diracMatrix] := j[a, ρ, b, ν, c]

δ /: j_[a___, μ_, b___, ν_, c___] δ[ν_, ρ_] /; FreeQ[j, diracMatrix] := j[a, μ, b, ρ, c]
δ /: j_[a___, μ_, b___, ν_, c___] δ[ρ_, ν_] /; FreeQ[j, diracMatrix] := j[a, μ, b, ρ, c]

δ /: δ[a_, b_]^2 := δ[a, a]
δ /: δ[a_, a_] := 8
```

Gamma Matrices

In[14]:=
```
diracMatrix /: diracMatrix[γ, μ_] ** diracMatrix[γ, μ_] /; ! SameQ[μ, 5] := d
diracMatrix /: diracMatrix[γ, μ_] ** diracMatrix[γ, ν_] ** diracMatrix[γ, μ_] /;
  ! SameQ[μ, 5] && ! SameQ[ν, 5] := (2 - d) diracMatrix[γ, ν]
diracMatrix /: diracMatrix[γ, μ_] ** diracMatrix[γ, ν_] ** diracMatrix[γ, λ_] **
   diracMatrix[γ, μ_] /; ! SameQ[μ, 5] && ! SameQ[ν, 5] && ! SameQ[λ, 5] := d * g[ν, λ]
diracMatrix /: diracMatrix[γ, μ_] ** diracMatrix[γ, ν_] **
   diracMatrix[γ, λ_] ** diracMatrix[γ, σ_] ** diracMatrix[γ, μ_] /;
  ! SameQ[μ, 5] && ! SameQ[ν, 5] && ! SameQ[λ, 5] && ! SameQ[σ, 5] :=
 (2 - d) diracMatrix[γ, ν] ** diracMatrix[γ, λ] ** diracMatrix[γ, σ]

diracMatrix /: diracMatrix[id] ** A_ /; FreeQ[A, _Complex] := A
diracMatrix /: A_ ** diracMatrix[id] /; FreeQ[A, _Complex] := A
diracMatrix /: A_ ** diracMatrix[id] ** B_ := A ** B

diracMatrix /: diracMatrix[γ, 5] ** diracMatrix[γ, 5] := diracMatrix[id]
tr /: tr[diracMatrix[γ, 5]] := 0
tr /: tr[diracMatrix[γ, 5] ** diracMatrix[γ, μ_] ** diracMatrix[γ, ν_]] /;
  ! SameQ[μ, 5] && ! SameQ[ν, 5] := 0
tr /: tr[diracMatrix[γ, 5] ** diracMatrix[γ, μ_] **
    diracMatrix[γ, ν_] ** diracMatrix[γ, λ_] ** diracMatrix[γ, σ_]] /;
  ! SameQ[μ, 5] && ! SameQ[ν, 5] && ! SameQ[λ, 5] && ! SameQ[σ, 5] := 4 * I * Levi[μ, ν, λ, σ]
tr /: tr[diracMatrix[id]] := d
tr /: tr[NonCommutativeMultiply[a : diracMatrix[γ, _] .., diracMatrix[γ, 5], b___]] /;
  ! MemberQ[{a}, diracMatrix[γ, 5]] :=
 ((-1) ^ (Length[{a}])) tr[diracMatrix[γ, 5] ** a ** b]
tr /: tr[NonCommutativeMultiply[a___, diracMatrix[γ, 5],
    b : diracMatrix[γ, _] .., diracMatrix[γ, 5], c___]] /;
  ! MemberQ[{b}, diracMatrix[γ, 5]] := ((-1) ^ (Length[{b}])) tr[a ** b ** c]
sigma[μ_, ν_] := (I / 2) *
  (diracMatrix[γ, μ] ** diracMatrix[γ, ν] - diracMatrix[γ, ν] ** diracMatrix[γ, μ])
```

Trace Identities

```
In[29]:= (*Define a new Trace*)
SetAttributes[tr, {Flat, OneIdentity}]
tr /: tr[x_ + y_] := tr[x] + tr[y]
tr /: tr[α_ ** A__] /; NumberQ[α] := α tr[A]

(*Trace Theorems*)
tr /: tr[A_] /; ! MemberQ[A, diracMatrix[γ, 5]] &&
    FreeQ[A, lam[__]] && FreeQ[A, m] && FreeQ[A, M] && FreeQ[A, _Complex] &&
    OddQ[Length[A]] && MemberQ[A, Repeated[diracMatrix[γ, _]]] := 0
tr /: tr[diracMatrix[γ, a_] ** diracMatrix[γ, b_]] /;
   ! IntegerQ[a] && ! IntegerQ[b] := 4 * g[a, b]

tr /: tr[x__] /; EvenQ[Length[x]] && Length[x] > 2 && ! MemberQ[x, diracMatrix[γ, 5]] &&
    MemberQ[x, Repeated[diracMatrix[γ, _]]] && FreeQ[x, lam[__]] &&
    FreeQ[x, _Complex] && FreeQ[x, momentum[__]] && FreeQ[x, m] && FreeQ[x, M] :=
  Sum[((-1) ^n) * g[Level[x[[1]], 1][[2]], Level[x[[n]], 1][[2]]] *
    tr[Drop[Drop[x, {1}], {n - 1}]], {n, 2, Length[x]}]

(* Extra Trace Fiddling*)
tr /: tr[A___ ** a_ * B_ ** C___] /; MemberQ[B, δ[_, _]] ||
    MemberQ[B, momentum[_, _]] || MemberQ[B, d] := B * tr[A ** a ** C]
tr /: tr[diracMatrix[γ, a_]] := 0
```

NonCommutativeMultiply

In[37]:=
```
Unprotect[NonCommutativeMultiply];
ClearAll[NonCommutativeMultiply]
NonCommutativeMultiply[a___, n_?NumericQ * c_, b___] := n a ** c ** b
NonCommutativeMultiply[a___, d * c_, b___] := d * a ** c ** b
NonCommutativeMultiply[a___, 0, b___] := 0
NonCommutativeMultiply[] := diracMatrix[id]
NonCommutativeMultiply /: NonCommutativeMultiply[A_] := A
NonCommutativeMultiply /: NonCommutativeMultiply[Z___, A___ / B__, X___] :=
 NonCommutativeMultiply[Z, A, X] / B

NonCommutativeMultiply /: A___ ** (gs * C__ ** E___) ** B___ := gs * A ** C ** E ** B
NonCommutativeMultiply /:
 NonCommutativeMultiply[A___, lam[a_, b_, c_], B__] := lam[a, b, c] A ** B
NonCommutativeMultiply /: NonCommutativeMultiply[A___, momentum[a_, b_], B___] :=
 momentum[a, b] A ** B
NonCommutativeMultiply /:
 NonCommutativeMultiply[A___, δ[a_, b_], B___] := δ[a, b] A ** B
NonCommutativeMultiply /:
 NonCommutativeMultiply[A___, g[a_, b_], B___] := g[a, b] A ** B

NonCommutativeMultiply /:
 NonCommutativeMultiply[A___, lam[a_, b_, c_] * C__, B___] := lam[a, b, c] A ** C ** B
NonCommutativeMultiply /: NonCommutativeMultiply[A___, δ[a_, b_] * C__, B___] :=
 δ[a, b] A ** C ** B
NonCommutativeMultiply /: NonCommutativeMultiply[A___, g[a_, b_] * C__, B___] :=
 g[a, b] A ** C ** B
NonCommutativeMultiply /: NonCommutativeMultiply[A___,
  Levi[a_, b_, c_, d_] * C__, B___] := Levi[a, b, c, d] A ** C ** B
NonCommutativeMultiply /: F___ ** (A_ + B_) ** C___ := F ** A ** C + F ** B ** C
NonCommutativeMultiply /: NonCommutativeMultiply[A__, B_, C___] /;
  MemberQ[{B}, g[_, _]] || MemberQ[{B}, lam[_, _, _]] || MemberQ[{B}, δ[_, _]] ||
   MemberQ[{B}, Levi[_, _, _, _]] := Times[B, NonCommutativeMultiply[A, C]]

NonCommutativeMultiply /: NonCommutativeMultiply[A___, Times[x_, C_], D___] /;
  SameQ[x, gs] || SameQ[x, m] || SameQ[x, M] || MemberQ[{x}, momentum[_, _]] :=
 Times[x, NonCommutativeMultiply[A, C, D]]

NonCommutativeMultiply /:
 NonCommutativeMultiply[A___, momentum[p_, μ_] diracMatrix[γ, μ_], B___] :=
 momentum[p, μ] * NonCommutativeMultiply[A, diracMatrix[γ, μ], B]

SetAttributes[NonCommutativeMultiply, {Flat, OneIdentity}]
Protect[NonCommutativeMultiply];
```

Four Vectors

In[60]:=
```
tr /: tr[A___ ** p_ ** B___] /; FreeQ[p, diracMatrix[γ, _]] &&
    FreeQ[p, lam[_]] && FreeQ[p, diracMatrix[id]] := p tr[A ** B]
tr /: tr[A___ * p_ * B___] /; FreeQ[p, diracMatrix[γ, _]] &&
    FreeQ[p, lam[_]] && FreeQ[p, diracMatrix[id]] := p tr[A ** B]

momentum /: momentum[a_ * p_, ν_] /; a ∈ Complexes := a * momentum[p, ν]
momentum /: momentum[p_, μ_] momentum[q_, μ_] := p.q
momentum /: momentum[p_, μ_]^2 := p.p
momentum /: momentum[p_ + q_, μ_] := momentum[p, μ] + momentum[q, μ]
```

Mass and Other Constants

In[66]:=
```
tr /: tr[(gs)^a_ * A___] := (gs)^a tr[A]
tr /: tr[A___ ** a_ ** B___] /;
   FreeQ[a, diracMatrix[γ, _]] && FreeQ[a, lam[_]] := a tr[A ** B]
tr /: tr[A___ ** (gs * C_ ** E___) ** B___] := gs * tr[A ** C ** E ** B ]
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply /: A___ ** m^a_ ** B___ := (m^a) A ** B
NonCommutativeMultiply /: A___ ** M^a_ ** B___ := (M^a) A ** B
NonCommutativeMultiply /: A___ ** gs ** B___ := gs A ** B
Protect[NonCommutativeMultiply];
```

Metric Identities

In[74]:=
```
g /: g[μ_, μ_] := d
g /: g[μ_, λ_] g[λ_, ν_] := g[μ, ν]
g /: g[μ_, λ_] g[ν_, λ_] := g[μ, ν]
g /: g[λ_, μ_] g[λ_, ν_] := g[μ, ν]
g /: g[μ_, ν_] momentum[p_, ν_]  := momentum[p, μ]
g /: g[μ_, ν_] momentum[p_, μ_] := momentum[p, ν]
g /: g[μ_, ν_] momentum[p_, ν_] momentum[k_, μ_] := momentum[p, ν] momentum[k, ν]
g /: g[μ_, ρ_] momentum[p_, ν_] momentum[k_, ρ_] := momentum[p, ν] momentum[k, μ]

g /: g[μ_, ν_] g[μ_, ν_] := g[μ, μ]
g /: g[μ_, ν_]^2 := g[μ, μ]

g /: g[μ_, ν_] NonCommutativeMultiply[diracMatrix[γ, ν_], A__] /; ! SameQ[ν, 5] :=
  NonCommutativeMultiply[diracMatrix[γ, μ], A]
```

Gell-Mann Matrices

In[85]:=
```
lam /: lam[a_, α1_, α4_] lam[b_, α4_, α1_] := tr[lam[a] ** lam[b]]
lam /: lam[a_, α1_, α2_] lam[b_, α2_, α3_] lam[c_, α3_, α1_] :=
 tr[lam[a] ** lam[b] ** lam[c]]
tr /: tr[A___ ** lam[a_, b_, c_] ** B___] := lam[a, b, c] tr[A ** B]

tr /: tr[lam[a_] ** lam[b_]] := 2 δ[a, b]
tr /: tr[lam[a_] ** lam[b_] ** lam[c_]] := 2 (dSym[a, b, c] + I * f[a, b, c])
tr /: tr[lam[a_] ** lam[b_] ** lam[c_] ** lam[d_]] :=
 (4 / 3) (δ[a, b] δ[c, d] - δ[a, c] δ[b, d] + δ[a, d] δ[b, c]) +
  2 (dSym[a, b, r] dSym[c, d, r] -
     dSym[a, c, r] dSym[d, b, r] + dSym[a, d, r] dSym[b, c, r]) +
  2 I (dSym[a, b, r] f[c, a, r] - dSym[a, c, r] f[a, b, r] + dSym[a, d, r] f[b, c, r])

lam /: lam[a_, α1_, α4_] ** diracMatrix[id] := lam[a, α1, α4]
lam /: diracMatrix[id] ** lam[a_, α1_, α4_] := lam[a, α1, α4]
```

Structure Constants

In[93]:=
```
f[a_, b_, c_] /; a ∈ Integers && b ∈ Integers && c ∈ Integers := Signature[{a, b, c}] *
  Piecewise[{{1, Sort[{a, b, c}][[3]] == 3}, {Sqrt[3] / 2, Sort[{a, b, c}][[3]] == 8},
    {-1 / 2, Sort[{a, b, c}] == {3, 6, 7} || Sort[{a, b, c}] == {1, 5, 6}},
    {1 / 2, Sort[{a, b, c}] == {1, 4, 7} || Sort[{a, b, c}] == {2, 4, 6} ||
      Sort[{a, b, c}] == {2, 5, 7} || Sort[{a, b, c}] == {3, 4, 5}}}, 0]
f /: f[a_, b_, c_] /; Signature[{a, b, c}] == -1 := Signature[{a, b, c}]
  f[Sort[{a, b, c}][[1]], Sort[{a, b, c}][[2]], Sort[{a, b, c}][[3]]]
(*Double Contraction*)
f /: f[a_, b_, c_] f[d_, b_, c_] := 3 δ[a, d]
f /: f[a_, b_, c_] f[a_, b_, d_] := 3 δ[c, d]
f /: f[a_, b_, c_] f[a_, c_, d_] := 3 δ[b, d]
(*Single Contraction*)
dSym /: dSym[a_, b_, c_] /; Signature[{a, b, c}] == 0 := 0
```

Levi-Civita Tensor

TARCER Code

In[104]:=
```
(*Import TARCER Modules*)
```

In[105]:=
```
<< "C:\\Users\\300075403.AD-UFV\\Documents\\Dropbox\\QCD
   Research\\2013 - QCD Research\\tarcer23.mx"
<< "F:\\Dropbox\\QCD Research\\2013 - QCD Research\\TARCER\\tarcer23.mx"
<< "/home/jason/Documents/TARCER/tarcer23.mx"
```

```
In[108]:= Tarce /: Tarce[x_ + y_, m__] := Tarce[x, m] + Tarce[y, m]
          Tarce /: Tarce[B___ * c_ * A___, p_, q_, k_, m__] /; FreeQ[c, q] && FreeQ[c, k] :=
           c * Tarce[B * A, p, q, k, m]
          Tarce /: Tarce[B___ * c_ * A__, p_, q_, k_, m__] /; FreeQ[c, q] && FreeQ[c, k] :=
           c * Tarce[B * A, p, q, k, m]
          Tarce /: Tarce[B___ * p_. p_ * A__, p_, q_, k_, m__] := (p.p) * Tarce[B * A, p, q, k, m]
          Tarce /: Tarce[A_ / c_, p_, q_, k_, m__] /; FreeQ[c, q] && FreeQ[c, k] :=
           (1 / c) * Tarce[A, p, q, k, m]

In[113]:= numeratorReplace /: numeratorReplace[A_ + B_, C__] :=
           numeratorReplace[A, C] + numeratorReplace[B, C]

In[114]:= numeratorReplace[A_, p_, q_, k_] :=
           Which[MemberQ[Numerator[A], (p - q).(p - q) | (q - p).(q - p), Infinity],
            (Numerator[A] /. (p - q).(p - q) | (q - p).(q - p) → p.p + q.q - 2 q.p) / Denominator[A],
            MemberQ[Numerator[A], (p + q).(p + q) | (q + p).(q + p), Infinity],
            (Numerator[A] /. (p + q).(p + q) | (q + p).(q + p) → p.p + q.q + 2 q.p) / Denominator[A],
            True, A]
```

```
In[115]:= Tarce /: Tarce[x_, p_, k1_, k2_, m1_, m2_, m3_, m4_, m5_] :=
  ((π) ^ (d)) (-1) ^ (Exponent[x, m1^2 - k1.k1] +
      Exponent[x, m2^2 - k2.k2] + Exponent[x, m3^2 - (p - k1).(p - k1)] +
      Exponent[x, m3^2 - (k1 - p).(k1 - p)] + Exponent[x, m4^2 - (-k2 + p).(-k2 + p)] +
      Exponent[x, m4^2 - (k2 - p).(k2 - p)] + Exponent[x, m5^2 - (-k2 + k1).(-k2 + k1)] +
      Exponent[x, m5^2 - (k2 - k1).(k2 - k1)]) TFI[d, p.p,
    {
     If[Exponent[x, k1.k1] > 0, Exponent[x, k1.k1], 0],
     If[Exponent[x, k2.k2] > 0, Exponent[x, k2.k2], 0],
     If[Exponent[x, p.k1] > 0, Exponent[x, p.k1],
      If[Exponent[x, k1.p] > 0, Exponent[x, k1.p], 0]],
     If[Exponent[x, p.k2] > 0, Exponent[x, p.k2],
      If[Exponent[x, k2.p] > 0, Exponent[x, k2.p], 0]],
     If[Exponent[x, k2.k1] > 0, Exponent[x, k2.k1],
      If[Exponent[x, k1.k2] > 0, Exponent[x, k1.k2], 0]]
    },
    {
     {If[Exponent[x, k1.k1 - m1^2] < 0, -Exponent[x, k1.k1 - m1^2],
       If[Exponent[x, m1^2 - k1.k1] < 0, -Exponent[x, m1^2 - k1.k1], 0]], m1},
     {If[Exponent[x, k2.k2 - m2^2] < 0, -Exponent[x, k2.k2 - m2^2],
       If[Exponent[x, m2^2 - k2.k2] < 0, -Exponent[x, m2^2 - k2.k2], 0]], m2},
     {If[Exponent[x, (p - k1).(p - k1) - m3^2] < 0,
       -Exponent[x, (p - k1).(p - k1) - m3^2],
       If[Exponent[x, (k1 - p).(k1 - p) - m3^2] < 0,
        -Exponent[x, (k1 - p).(k1 - p) - m3^2],
        If[Exponent[x, m3^2 - (p - k1).(p - k1)] < 0,
         -Exponent[x, m3^2 - (p - k1).(p - k1)],
         If[Exponent[x, m3^2 - (k1 - p).(k1 - p)] < 0,
          -Exponent[x, m3^2 - (k1 - p).(k1 - p)], 0]]]], m3},
     {If[Exponent[x, (-k2 + p).(-k2 + p) - m4^2] < 0,
       -Exponent[x, (-k2 + p).(-k2 + p) - m4^2],
       If[Exponent[x, (k2 - p).(k2 - p) - m4^2] < 0,
        -Exponent[x, (k2 - p).(k2 - p) - m4^2],
        If[Exponent[x, m4^2 - (-k2 + p).(-k2 + p)] < 0,
         -Exponent[x, m4^2 - (-k2 + p).(-k2 + p)],
         If[Exponent[x, m4^2 - (k2 - p).(k2 - p)] < 0,
          -Exponent[x, m4^2 - (k2 - p).(k2 - p)], 0]]]], m4},
     {If[Exponent[x, (-k2 + k1).(-k2 + k1) - m5^2] < 0,
       -Exponent[x, (-k2 + k1).(-k2 + k1) - m5^2],
       If[Exponent[x, (k2 - k1).(k2 - k1) - m5^2] < 0,
        -Exponent[x, (k2 - k1).(k2 - k1) - m5^2],
        If[Exponent[x, m5^2 - (-k2 + k1).(-k2 + k1)] < 0,
         -Exponent[x, m5^2 - (-k2 + k1).(-k2 + k1)],
         If[Exponent[x, m5^2 - (k2 - k1).(k2 - k1)] < 0,
          -Exponent[x, m5^2 - (k2 - k1).(k2 - k1)], 0]]]], m5}
    }
   ]

In[154]:= MassiveTadpole[k_, M_] := (1 / (k.k - M^2)) / (π^(d / 2) TAI[d, 0, {{1, M}}])
```

Rules

Simplifying notation

```
In[116]:= momentumConvert := {momentum[p_, μ_] → p[μ], p.p → p^2, q.q → q^2, k.k → k^2}
momentumUnConvert := {p_[μ_] → momentum[p, μ], p^2 → p.p, q^2 → q.q, k^2 → k.k}
toEpsilon := {d → 4 + 2 ϵ}
```

Integrals

```
In[119]:= (*result from Davydychev & Boos, Theor. Math. Phys. (1991)*)
evalInt :=
 {TJI[d, Dot[p_, p_] | Power[p_, 2], List[List[ν_, M_], List[1, 0], List[1, 0]]] :→
   (-1) (M^2) ^ (2 + 2 ϵ - ν) (-1) ^ (-ν)
    ((Gamma[1 + ϵ] Gamma[ν - 2 - 2 ϵ] Gamma[1 + ϵ] Gamma[-ϵ]) / (Gamma[2 + ϵ] Gamma[ν])) *
    (HoldForm[HypergeometricPFQ[{-ϵ, ν - 2 - 2 ϵ}, {2 + ϵ}, (p / M) ^2]]),
  TBI[d, Dot[q_, q_] | Power[q_, 2], List[List[β_, M_], List[α_, 0]]] :>
   I * (-1) ^ (-α - β) * (M^2) ^ (d / 2 - α - β) *
    ((Gamma[d / 2 - β] Gamma[α + β - d / 2]) / (Gamma[d / 2] Gamma[α])) *
    HypergeometricPFQ[{β, α + β - d / 2}, {d / 2}, q^2 / M^2],
  TAI[d, 0, {{1, M_}}] :→ -I * (M^2) ^ (d / 2 - 1) * Gamma[1 - d / 2]}
evalDerivative :=
 {D[TJI[d, Power[p_, 2], List[List[ν_, M_], List[μ_, m_], List[1, 0]]], {m_, 1}] :→
   2 μ * m (TJI[d, Power[p, 2], List[List[ν, M], List[μ + 1, m], List[1, 0]]]) ,
  D[TJI[d, Power[p_, 2], List[List[ν_, M_], List[μ_, m_], List[1, 0]]], {m_, 2}] :→
   μ (TJI[d, Power[p, 2], List[List[ν, M], List[1 + μ, m], List[1, 0]]] +
      m (TJI[d, Power[p, 2], List[List[ν, M], List[μ + 2, m], List[1, 0]]])) }
```

Formatting

Formulas

```
In[122]:= Unprotect[KroneckerDelta];
Format[δ[a_, b_], StandardForm] := Superscript["δ", {a, b}]
Protect[KroneckerDelta];
Format[f[a_, b_, c_], StandardForm] := Superscript["f", {a, b, c}]
Format[momentum[p_, μ_], StandardForm] := Subscript[Style[p, Bold], μ]
Format[lam[a_, α1_, α2_], StandardForm] := Subscript[Superscript["λ", a], {α1, α2}]
Format[Levi[a_, b_, c_, d_], StandardForm] := Superscript["ϵ", {a, b, c, d}]
Format[g[a_, b_], StandardForm] := Subscript["g", {a, b}]
Format[diracMatrix[γ, b_], StandardForm] := Superscript[Style["γ", Bold], {b}]
```

Contributions

In[131]:=
```
(*Functions describing the contributions of each vertice and propogator in QCD*)
(*Always enter in the order: Lorentz indices, color indices,
momentums, lambda matrix entries,and free indices (for summation) *)
```

Projection Operators (Vector/Scalar)

In[132]:=
```
GBVec[lorentz1_, lorentz2_, color1_, color2_, m1_] := 1 / ((m1.m1) * (d - 1))
  (momentum[m1, lorentz1] momentum[m1, lorentz2] - m1.m1 * g[lorentz1, lorentz2])
GBSca[lorentz1_, lorentz2_, color1_, color2_, m1_] :=
 momentum[m1, lorentz1] momentum[m1, lorentz2]
```

Propogators

In[134]:=
```
GluProp[lorentz1_, lorentz2_, color1_, color2_, m1_] :=
 -I * diracMatrix[id] ** g[lorentz1, lorentz2] * δ[color1, color2] / (m1.m1)
FermProp[m_, m1_, lamEntry1_, lamEntry2_, free_] :=
 I * diracMatrix[id] ** δ[lamEntry1, lamEntry2] *
  (momentum[m1, free] ** diracMatrix[γ, free] + m) / (m1.m1 - m^2)
FermPropD[m_, m1_, free_] := (momentum[m1, free] ** diracMatrix[γ, free] + m) /
  (m1.m1 - m^2) (* No Imaginary Factor - for derivative work*)
GhostProp[color1_, color2_, m1_] := I * diracMatrix[id] ** δ[color1, color2] / (m1.m1)

DFermProp[m_, m1_, free1_, free2_, derivindex1_] :=
 -((momentum[m1, free1] ** diracMatrix[γ, free1] - m) / (m1.m1 - m^2)) ** diracMatrix[γ,
    derivindex1] ** ((momentum[m1, free2] ** diracMatrix[γ, free2] - m) / (m1.m1 - m^2))
DDFermProp[m_, m1_, free1_, free2_, free3_, derivindex1_, derivindex2_] :=
 (((momentum[m1, free1] ** diracMatrix[γ, free1] + m) / (m1.m1 - m^2)) **
    diracMatrix[γ, derivindex2] **
    ((momentum[m1, free2] ** diracMatrix[γ, free2] + m) / (m1.m1 - m^2)) **
    diracMatrix[γ, derivindex1] **
    ((momentum[m1, free3] ** diracMatrix[γ, free3] + m) / (m1.m1 - m^2)) +
   ((momentum[m1, free1] ** diracMatrix[γ, free1] + m) / (m1.m1 - m^2)) **
    diracMatrix[γ, derivindex1] **
    ((momentum[m1, free2] ** diracMatrix[γ, free2] + m) / (m1.m1 - m^2)) **
    diracMatrix[γ, derivindex2] **
    ((momentum[m1, free3] ** diracMatrix[γ, free3] + m) / (m1.m1 - m^2)))
```

Vertices

```
In[140]:= GluVert3[lorentz1_, lorentz2_, lorentz3_, color1_, color2_, color3_, m1_, m2_, m3_] :=
  gs * diracMatrix[id] ** f[color1, color2, color3] *
   (g[lorentz1, lorentz2] (momentum[m1, lorentz3] - momentum[m2, lorentz3]) +
     g[lorentz2, lorentz3] (momentum[m2, lorentz1] - momentum[m3, lorentz1]) +
     g[lorentz3, lorentz1] (momentum[m3, lorentz2] - momentum[m1, lorentz2]))
GluVert4[lorentz1_, lorentz2_, lorentz3_, lorentz4_, color1_,
   color2_, color3_, color4_, free_] :=
  -I * (gs) ^2 * diracMatrix[id] ** (f[color1, color2, free] f[color3, color4, free]
        (g[lorentz1, lorentz3] g[lorentz2, lorentz4] -
          g[lorentz1, lorentz4] g[lorentz2, lorentz3]) + f[color1, color4, free]
        f[color2, color3, free] (g[lorentz1, lorentz2] g[lorentz3, lorentz4] -
          g[lorentz1, lorentz3] g[lorentz2, lorentz4]) +
      f[color1, color3, free] f[color4, color2, free] (g[lorentz1, lorentz4]
          g[lorentz2, lorentz3] - g[lorentz1, lorentz2] g[lorentz3, lorentz4]))
FermVert[lorentz1_, color1_, lamEntry1_, lamEntry2_] := (I / 2) * gs *
   diracMatrix[id] ** lam[color1, lamEntry1, lamEntry2] ** diracMatrix[γ, lorentz1]
GhostVert[lorentz1_, color1_, color2_, color3_, m1_] :=
  -gs * diracMatrix[id] ** f[color1, color2, color3] * momentum[m1, lorentz1]

G5Current1[lorentz1_, lorentz2_, color1_,
   lamEntry1_, lamEntry2_, momentum1_, free1_, free2_] :=
  (I / 2) * gs * diracMatrix[id] ** Levi[lorentz1, lorentz2, free1, free2] **
    lam[color1, lamEntry1, lamEntry2] ** momentum[momentum1, free1] **
    diracMatrix[γ, free2] ** diracMatrix[γ, 5]
Current1[lorentz1_, lorentz2_, color1_, lamEntry1_,
   lamEntry2_, momentum1_, free1_, free2_] :=
  (I / 2) * gs * diracMatrix[id] ** Levi[lorentz1, lorentz2, free1, free2] ** lam[color1,
     lamEntry1, lamEntry2] ** momentum[momentum1, free1] ** diracMatrix[γ, free2]
G5Current2[lorentz1_, lorentz2_, color1_, lamEntry1_,
   lamEntry2_, momentum1_, free1_, free2_] :=
  (I / 2) * gs * diracMatrix[id] ** lam[color1, lamEntry1, lamEntry2] **
    (g[lorentz2, lorentz1] momentum[momentum1, free2] - g[lorentz2, free2]
       momentum[momentum1, lorentz1]) ** diracMatrix[γ, free2] ** diracMatrix[γ, 5]
Current2[lorentz1_, lorentz2_, color1_, lamEntry1_, lamEntry2_, momentum1_, free1_,
   free2_] := (I / 2) * gs * diracMatrix[id] ** lam[color1, lamEntry1, lamEntry2] **
    (g[lorentz2, lorentz1] momentum[momentum1, free2] -
      g[lorentz2, free2] momentum[momentum1, lorentz1]) ** diracMatrix[γ, free2]
```

Non-Perturbative Contributions

```
In[148]:= GVacuumFactor[lorentz1_, lorentz2_, lorentz3_, lorentz4_] :=
  (1 / (d^2 - d)) (g[lorentz1, lorentz3] g[lorentz2, lorentz4] -
     g[lorentz1, lorentz4] g[lorentz2, lorentz3]) vev[GG]
```

```
In[149]:= DDGVacuumFactor[lorentz1_, lorentz2_,
            lorentz3_, lorentz4_, derivindex1_, derivindex2_] :=
           (2 / ((d + 2) d (d - 2))) vev[gfGGG] g[derivindex1, derivindex2] *
             (g[lorentz1, lorentz3] g[lorentz2, lorentz4] -
               g[lorentz1, lorentz4] g[lorentz2, lorentz3]) + (1 / ((d + 2) d (d - 2))) vev[gfGGG]
             (g[derivindex2, lorentz3] (g[derivindex1, lorentz1] g[lorentz2, lorentz4] -
                  g[derivindex1, lorentz2] g[lorentz1, lorentz4])
               - g[derivindex2, lorentz4] (g[derivindex1, lorentz1] g[lorentz2, lorentz3] -
                  g[derivindex1, lorentz2] g[lorentz1, lorentz3])) +
            (-3 / ((d + 2) d (d - 1) (d - 2))) vev[gfGGG] (g[derivindex1, lorentz3]
                (g[derivindex2, lorentz1] g[lorentz2, lorentz4] -
                  g[derivindex2, lorentz2] g[lorentz1, lorentz4])
               - g[derivindex1, lorentz4] (g[derivindex2, lorentz1] g[lorentz2, lorentz3] -
                  g[derivindex2, lorentz2] g[lorentz1, lorentz3]))

In[150]:= GTwidVacuumFactor[lorentz1_, lorentz2_, lorentz3_, lorentz4_, free1_,
            free2_, free3_, free4_] := (1 / 4) * Levi[lorentz1, lorentz2, free1, free2]
            Levi[lorentz3, lorentz4, free3, free4] GVacuumFactor[free1, free2, free3, free4]

In[151]:= DDGTwidVacuumFactor[lorentz1_, lorentz2_, lorentz3_, lorentz4_,
            derivindex1_, derivindex2_, free1_, free2_, free3_, free4_] := (1 / 4) *
            Levi[lorentz1, lorentz2, free1, free2] Levi[lorentz3, lorentz4, free3, free4]
            DDGVacuumFactor[free1, free2, free3, free4, derivindex1, derivindex2]

In[152]:= (* Indices ordered as read off of Gluon Field Strength Tensors *)
          ThreeVacuumFactor[lorentz1_, lorentz2_, lorentz3_,
            lorentz4_, lorentz5_, lorentz6_] := (vev[gfGGG] / (d (d - 1) (d - 2))) *
            (g[lorentz2, lorentz3] g[lorentz4, lorentz5] g[lorentz6, lorentz1] -
              g[lorentz1, lorentz3] g[lorentz4, lorentz5] g[lorentz6, lorentz2] +
              g[lorentz1, lorentz4] g[lorentz3, lorentz5] g[lorentz6, lorentz2] -
              g[lorentz2, lorentz4] g[lorentz3, lorentz5] g[lorentz6, lorentz1] +
              g[lorentz2, lorentz4] g[lorentz3, lorentz6] g[lorentz5, lorentz1] -
              g[lorentz1, lorentz4] g[lorentz3, lorentz6] g[lorentz5, lorentz2] +
              g[lorentz1, lorentz3] g[lorentz4, lorentz6] g[lorentz5, lorentz2] -
              g[lorentz2, lorentz3] g[lorentz4, lorentz6] g[lorentz5, lorentz1])

In[153]:= TwidThreeVacuumFactor[lorentz1_, lorentz2_, lorentz3_, lorentz4_,
            lorentz5_, lorentz6_, free1_, free2_, free3_, free4_] := (1 / 4)
            Levi[lorentz1, lorentz2, free1, free2] Levi[lorentz3, lorentz4, free3, free4]
            ThreeVacuumFactor[free1, free2, free3, free4, lorentz5, lorentz6]
```